Exploitation as the Unequal Exchange of Labour: An Axiomatic Approach

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Abstract

In subsistence economies with general convex technology and rational optimising agents, a new, axiomatic approach is developed, which allows an explicit analysis of the core positive and normative intuitions behind the concept of exploitation. Two main new axioms, called Labour Exploitation in Subsistence Economies and Relational Exploitation are presented together with the continuity, independence, and replication invariance properties, and it is proved

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that they uniquely characterise a definition of exploitation conceptually related to the so-called “New-Interpretation” (Duménil, 1980; Foley, 1982; Duménil et al., 2009), which focuses on the unequal distribution of (and control over) social labour, and on individual well-being freedom and the self-realisation of men. Then, the main results of Roemer’s (1982, 1988) classical approach and all the crucial insights of exploitation theory are generalised, proving that every agent’s class and exploitation status emerges in the competitive equilibrium, that there is a correspondence between an agent’s class and exploitation status, and that the existence of exploitation is inherently linked to the existence of positive profits.

**JEL:** D63 (Equity, Justice, Inequality, and Other Normative Criteria and Measurement); D51 (General Equilibrium: Exchange and Production Economies); C62 (Existence and Stability Conditions of Equilibrium); B51 (Socialist; Marxist, Sraffian).

**Keywords:** Justice, Exploitation, Class, Convex Economies.
1 Introduction

The notion of exploitation is prominent in the social sciences and in political discourse. It is central in a number of debates, ranging from analyses of labour relations, especially focusing on the weakest segments of the labour force, such as children, women, and migrants (see, e.g., ILO, 2005; 2005a; 2006); to controversies on drug-testing and on the price of life-saving drugs, especially in developing countries;\(^1\) to ethical issues arising in surrogate motherhood (see, e.g., Field, 1989; Wood, 1995). The concept of exploitation is also central in the politics of the Left. In the 2007 programme of the German Social Democratic Party, for example, the very first paragraph advocates a society ‘free from poverty, exploitation, and fear’ (SPD, 2007, p.3), and the fight against exploitation is repeatedly indicated as a priority for the biggest party of the European Left. The notion of exploitation is arguably the cornerstone of Marxist social theory, but it is also extensively discussed in normative theory and political philosophy (see, e.g., Wertheimer, 1996; Wolff, 1999; Bigwood, 2003; and Sample, 2003).\(^2\) Yet, there is little agreement concerning even the most basic features of exploitative relations, and both the definition of exploitation and its normative content are highly controversial.\(^3\)

In general, agent A exploits agent B if and only if A takes unfair advantage of B. Despite its intuitive appeal, this definition leaves two major issues in need of precise specification from a normative perspective, namely the source of the unfairness and the structure of the relationship between \(A\) and \(B\) that allows \(A\) to take advantage of \(B\). There is considerable debate in the economic and philosophical literature concerning both issues. At one extreme, in his seminal theory of exploitation, John Roemer (1982, 1988) argues that exploitation is a purely distributive concept which makes no ref-

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\(^1\)In a section devoted to ‘Ethical Issues,’ the Investigation Committee on the clinical trial of the drug ‘Trovan’ conducted by Pfizer in 1996 in Kano (Nigeria) argue that ‘Compensations to the participants were minimal or non existent, as such a clear case of exploitation of the ignorant was established’ (Federal Ministry of Health of Nigeria, 2001, p.88).

\(^2\)The notion of exploitation is relevant, for example, in Lockean or Neo-Lockean political philosophy, whereby exploitation occurs if the principle of Just Acquisition of unowned land is violated by State intervention; or in Neoclassical economic philosophy, whereby exploitation occurs in non-competitive markets if the distributive principle of marginal productivity is violated.

\(^3\)For a review of some of the debates in exploitation theory, see Nielsen and Ware (1997).
reference to the interaction between agents and identifies an injustice stemming from an unequal distribution of assets. At the other extreme, contra Roemer, various authors deny that exploitation involves a distributive injustice and claim that the moral force of exploitation derives entirely from the objectionable features of the interaction between agents (e.g., Wolff, 1999; Wood, 2004).

As shown by Veneziani (2008), and as acknowledged by Roemer himself in later contributions, Roemer’s claim that exploitation is a purely distributive concept is not convincing. Some notion of unequal power, or dominance, is arguably crucial in any theory of exploitation and the positive and normative analysis of exploitative relations involves some consideration of the way in which A and B interact. It seems, however, equally reductive to assume that inequalities deriving from exploitative relations between agents are immaterial in the judgment of exploitation. As forcefully argued by Warren (1997, p.63), “exploitation involves inequality on both ends of exchange: inequality defining the context of the exchange (that is, [differential ownership of productive assets]) and inequality defining the outcome.” So, although power, force, or dominance, are arguably crucial elements of exploitation theory, the analytical focus on this paper is on the source of the unfairness of exploitative relations, and on the injustice involved in the concept of exploitation within economic relations.

More specifically, this paper analyses the theory of exploitation as an unequal exchange (hereafter, UE) of labour, according to which exploitative relations are characterised by a difference between the hours of labour that an individual provides and the hours of labour necessary to produce commodities that she can purchase with her income. There are at least two reasons to focus on labour as the measure of the injustice of exploitative relations. First, in a number of crucial economic interactions, the notion of exploitation seems inextricably linked with some form of labour exchange. Second,

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4 For a thorough analysis of Roemer’s distributive approach, see Skillman (1995). Skillman (1995) argues that Roemer’s approach is consistent with Marx’s own account.

5 In fact, Yoshihara (1998) showed that exploitative status is linked with the degree of labour-discipline, which reflects power relations in capitalist economies.

6 In this perspective, “it is not unequal power itself that is supposed objectionable, but rather the fact that one person gains unjustly through the exercise of power (whether coercive or uncoercive) over another” (Warren, 1997, p.62). According to Warren, the relevant outcome inequality concerns indeed the unequal performance of labor, as suggested also in this paper.

7 Despite his initial criticism of the UE definition, Roemer has later acknowledged that
the UE definition of exploitation captures some inequalities in the distribution of material well-being and free hours that are - at least *prima facie* - of normative relevance. As shown in this paper, for example, it is possible to significantly generalise the so-called *Class-Exploitation Correspondence Principle* (hereafter, CECP; see Roemer, 1982, 1988), according to which in a private-ownership economy with positive profits, class and UE exploitation status are strictly related, and they accurately reflect an unequal distribution of assets. That is, in equilibrium, the wealthiest agents emerge as exploiters, and members of the capitalist class, whereas poor agents are exploited, and members of the working class. As in standard Marxist theory, then, exploitative relations are relevant in that they reflect unequal opportunities of life options, due to unequal access to productive assets.

Interestingly, however, the UE concept of exploitation can also be seen as capturing inequalities in the distribution of *well-being freedom*. In fact, as argued by Rawls (1971) and Sen (1985a, 1985b), among the others, an individual’s well-being freedom captures her ability to pursue the life she values. There are two crucial factors that determine the degree of individual’s well-being freedom, or self-realisation: one is the amount of income she can spend to purchase the commodities necessary to achieve her goals, and the other is the amount of time she has to sacrifice as labour supply in order to purchase such commodities. Then, a measure of UE exploitation can represent the degree of well-being freedom, or indeed *un*freedom, of the agent, since it can be taken as an index of the relative attainment of these two goods by using labour time as the *numéraire*: if an agent gains from the unequal exchange of labour, then she is exploiting the free hours which some other agents sacrificed as labour supply for the production of the commodities.

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8Interestingly, this notion of freedom is conceptually related to the Marxian notion of *self-realisation*, and one can argue that both commodities and free time are essential for an individual’s self-realisation.

9In the Rawls-Sen theory, inequalities in the distribution of well-being freedom are formulated as inequalities of capabilities. The resource allocation problem, in terms of equality of capability, is explicitly analysed in Gotoh and Yoshihara (2003).

10In this view, labour only yields disutility and it reduces the possibility to self-realise. To assume away completely the possibility that work itself may be a source of well-being freedom may be unrealistic, but it is appropriate at the level of analysis of this paper, and it is consistent with a Marxian analysis of capitalist relations of production.
ties she can purchase, whereas if she suffers from the unequal exchange of labour, she is exploited in that some of the free hours she sacrificed as labour supply to purchase the commodities are appropriated by somebody else.

Granting the normative relevance of the unequal exchange of labour, there are many possible ways of rigorously specifying the concept of UE exploitation and a number of alternative definitions have indeed been proposed in the literature (for a thorough discussion, see Yoshihara, 2010). This paper provides the first rigorous axiomatic analysis of UE exploitation: this is a completely new approach to exploitation theory and it provides a fully general framework to compare the most important definitions of exploitation discussed in the literature. An axiomatic approach was long overdue in exploitation theory, where the proposal of alternative definitions have sometimes appeared as a painful process of adjustment of the theory to the various counterexamples and formal exceptions found in the literature. The definitions of exploitation thus constructed have progressively lost the intuitive appeal, the normative relevance, and even the connection with the actual, observed variables emerging from a competitive mechanism. By taking an axiomatic approach, this paper suggests to start from first principles, thus explicitly discussing the normative intuitions behind exploitation theory. Therefore, the approach proposed in this paper, and the analysis developed below should be interesting for all exploitation theorists, and indeed for all social scientists and political philosophers, even if the specific axioms proposed may be deemed unsatisfactory.

To be precise, this paper analyses exploitation theory in a class of convex subsistence economies which generalise Roemer (1982, 1988). In this class of economies, each producer can use a general convex technology and is assumed to minimize her labour supply under the constraint that she has to earn enough to purchase a subsistence vector. First of all, a domain axiom is introduced, called Labour Exploitation in Subsistence Economies (hereafter, LES), which restricts the way in which the sets of exploiters and exploited agents should be identified in equilibrium. This axiom is taken as the minimal necessary condition to stipulate the normative intuitions behind exploitation theory, and it is shown that all the main definitions of exploitation proposed in the literature (see, for example, Morishima, 1974; Roemer, 1982; Yoshihara, 2010) do satisfy LES. Then, Theorems 1 and 2 provide a significant generalisation of Roemer’s (1982, 1988) celebrated results: they derive the equilibrium class and exploitation structures of a general convex cone subsistence economy with optimising agents for a whole class of definitions of
exploitation satisfying LES. Further, Theorems 3 and 4 derive the necessary and sufficient conditions under which, for a whole class of definitions of exploitation satisfying LES, and for any convex subsistence economy, the CECP holds and the existence of exploitation is synonymous with the existence of positive profits, respectively. These results are theoretically relevant because, as argued by Roemer (1982), although they are formally derived as theorems, their epistemological status is as postulates: any definition of exploitation should preserve them.

Then, four additional axioms are introduced: firstly, two axioms are introduced to capture the continuity and the replication invariance properties of UE exploitation. Secondly, the next axiom requires that the definition of the status of exploiter or exploited agent should be independent of irrelevant factors, where the irrelevant factor for defining UE exploitation is the distribution of productive endowments in the economy. The fourth axiom captures the relational nature of exploitative relations by ruling out the possibility that there exists exploiters without any agent being exploited, and vice versa. Interestingly, although almost all the formulations of UE exploitation discussed in the literature satisfy LES, the continuity, the replication invariance, and the independence axioms, Theorem 5 proves that a generalised version of the so-called “New Interpretation” (Duménil, 1980; Foley, 1982; Duménil at el., 2009) is the unique definition of exploitation that satisfies all axioms. As a corollary, it follows that the generalised “New Interpretation” is the unique formulation of UE exploitation that satisfies the four axioms and under which the CECP holds in the class of general convex cone subsistence economies.

There are two main reasons to focus on static subsistence economies. First of all, the analysis of subsistence economies with a labour market is theoretically crucial in that they provide the simplest institutional framework in which exploitation arises. In particular, in order to analyse the distributive issues related to exploitation, it is appropriate to abstract from the role that exploitation plays in the accumulation process.11 The model of a subsistence economy may not be a realistic representation of ‘actual economies,’ but it is

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11It is also worth noting that the subsistence vector can also be interpreted as a social reference bundle of commodities, which represents a decent living standard, rather than as a consumption bundle necessary for survival. In this case, once the decent living standard is reached, each agent is free to determine how to use her spare time, if any. Then, the existence of UE exploitation represents unequal allocations of free hours among agents, given that everyone reaches the decent living standard, which straightforwardly implies unequal opportunities for self-realisation and well-being freedom.
an abstract model suitable to illustrate some of the essential characteristics of market economies with private ownership of productive assets. Besides, from a theoretical viewpoint, one may argue that the distinctive characteristic of capitalist economies is the existence of a labour market, in which labour is exchanged as a commodity, rather than accumulation and growth. Second, from a formal viewpoint, subsistence economies provide a simple framework sufficient to identify the proper definition of UE exploitation axiomatically. Indeed, as argued in detail below, the main conclusion of Theorem 5 can be generalised to the universal domain of convex economies, which contains not only the domain of subsistence economies studied herein, but also the domains of accumulation economies (see Yoshihara, 2010) and dynamic economies along the lines, e.g., of Veneziani (2007, 2008) and Veneziani and Yoshihara (2010).

The rest of the paper is organised as follows. In section 2, the model of a general convex subsistence economy is set up. In section 3, the notions of exploitation and classes are defined and axiom LES is presented. It is then shown that all the most important definitions of exploitation presented in the literature satisfy LES. The complete class and exploitation structures of the economy for the class of definitions of exploitation satisfying LES are derived. In section 4, the necessary and sufficient conditions for the CECP to hold for a class of definitions of exploitation satisfying LES are derived. In Section 5 the four additional axioms, called Continuity, Replication Invariance, Independence, and Relational Exploitation are presented and the main characterisation result of the paper is derived. Section 6 concludes and the existence of a general equilibrium is proved in Appendix 1, whereas all the proofs of the formal results are in Appendix 2.

2 A Model of General Convex Subsistence Economies

Let $P$ be the production set. $P$ has elements of the form $\alpha = (-\alpha_0, -\underline{\alpha}, \bar{\alpha})$ where $\alpha_0 \in \mathbb{R}_+, \underline{\alpha} \in \mathbb{R}_+^m$, and $\bar{\alpha} \in \mathbb{R}_+^m$. Thus, elements of $P$ are vectors in $\mathbb{R}^{2m+1}$. The first component, $-\alpha_0$, is the direct labour input of the process $\alpha$; the next $m$ components, $-\underline{\alpha}$, are the inputs of goods used in the process; and the last $m$ components, $\bar{\alpha}$, are the outputs of the $m$ goods from the process. The net output vector arising from $\alpha$ is denoted as $\hat{\alpha} \equiv \bar{\alpha} - \underline{\alpha}$. The
set $P$ is assumed to be a closed convex cone containing the origin in $\mathbb{R}^{2m+1}$. Moreover, let $0 \in \mathbb{R}^m$ be such that $0 = (0, ..., 0)'$: it is assumed that\textsuperscript{12} 

A1. $\forall \alpha \in P$ s.t. $\alpha_0 \geq 0$ and $\alpha \geq 0$, $[\alpha \geq 0 \Rightarrow \alpha_0 > 0]$. 

A2. $\forall c \in \mathbb{R}_+^m$, $\exists \alpha \in P$ s.t. $\alpha \geq c$. 

A3. $\forall \alpha \in P$, $\forall (-\alpha', \bar{\alpha'}) \in \mathbb{R}_-^m \times \mathbb{R}_+^m$, $[(\alpha', \bar{\alpha'}) \leq (-\alpha, \bar{\alpha}) \Rightarrow (\alpha_0, -\alpha', \bar{\alpha'}) \in P]$. 

A1 implies that labour is indispensable to produce any non-negative output vector; A2 states that any non-negative commodity vector is producible as a net output; and A3 is a free disposal condition, which states that, given any feasible production process $\alpha$, any vector producing (weakly) less net output than $\alpha$ is also feasible using the same amount of labour as $\alpha$ itself.

Given $P$, it is possible to define the set of net output vectors that can be produced using exactly $l$ units of labour, denoted as $\hat{P}(\alpha_0 = l)$. Formally:

$$\hat{P}(\alpha_0 = l) \equiv \{\hat{\alpha} \in \mathbb{R}^m \mid \exists \alpha = (-l, -\alpha, \bar{\alpha}) \in P \text{ s.t. } \alpha - \alpha \geq \hat{\alpha}\}.$$ 

Finally, for any set $X \subseteq \mathbb{R}^m$, the boundary of $X$ is defined as $\partial X \equiv \{x \in X \mid \exists x' \in X \text{ s.t. } x' > x\}$, and $coX$ is the convex hull of the set $X$.

Consider a generalisation of Roemer’s (1982) subsistence economy. Let $N$ be the set of agents, with generic element $\nu$. All agents $\nu \in N$ have access to the same technology $P$, but they possess different endowments $\omega^\nu$, whose distribution in the economy is given by $(\omega^\nu)_{\nu \in N} \in \mathbb{R}_+^{Nm}$. An agent $\nu \in N$ endowed with $\omega^\nu$ can engage in three types of economic activity: she can sell her labour power $\gamma^\nu_0$, she can hire others to operate $\beta^\nu = (-\beta^\nu_0, -\beta^\nu, \bar{\beta}^\nu) \in P$, or she can work on her own operating $\alpha^\nu = (-\alpha^\nu_0, -\alpha^\nu, \bar{\alpha}^\nu) \in P$. Given a price vector $p \in \mathbb{R}^m$ and a nominal wage rate $w$, it is assumed that each agent chooses her activities, $\alpha^\nu$, $\beta^\nu$, and $\gamma^\nu_0$, in order to minimise the labour she expends subject to earning enough income to purchase a subsistence vector of commodities $b \in \mathbb{R}_+^m$. Moreover, she must be able to lay out in advance the operating costs for the activities she chooses to operate, either with her own labour or with hired labour, using her wealth, and she cannot work more than her labour endowment.

Formally, given $(p, w)$, every agent chooses $(\alpha^\nu, \beta^\nu, \gamma^\nu_0)$ in order to solve program $MP^\nu$:

$$\min \alpha^\nu_0 + \gamma^\nu_0$$

$\textsuperscript{12}$For all vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, $x \geq y$ if and only if $x_i \geq y_i$ ($i = 1, \ldots, m$); $x \geq y$ if and only if $x \geq y$ and $x \neq y$; $x > y$ if and only if $x > y$ and $x \neq y$ ($i = 1, \ldots, m$).
subject to
\[ p(\bar{\alpha} - \bar{\alpha}') + [p \left( \bar{\beta} - \bar{\beta}' \right) - w\beta_0] + [w\gamma_0] \geq pb, \]
\[ p\alpha' + p\beta' \leq p\omega', \]
\[ \alpha_0 + \gamma_0 \leq 1. \]

Given \((p, w)\), let \(A'\) be the set of actions \((\alpha', \beta'; \gamma_0')\) which solve \(\nu\)'s minimisation problem \(MP'\) at prices \((p, w)\). Following Roemer (1982), and unlike in the mainstream approach, \(MP'\) explicitly incorporates the simultaneous role of economic actors as consumers and producers (so that no separate consideration of firms is necessary) and the time structure of the production process. It is thus assumed that, at the beginning of the period, agents need to lay out in advance the capital needed for production and they can do so only using their own wealth.\(^{13}\) Production then takes place and the proceeds can be used to finance consumption and the reproduction of the initial wealth at the end of the period.\(^{14}\) Finally, as shown below, although agents are not assumed to maximise profits, profit maximisation is a corollary of \(MP'\).

Let a convex cone subsistence economy be given by a list \(E = \langle N; (P, b); (\omega\nu)\rangle\). Let \(E\) denote the set of all convex cone subsistence economies. Based on Roemer (1982), the equilibrium notion for an economy \(E \in E\) can be defined.

**Definition 1:** A reproducible solution (RS) for an economy \(E \in E\) is a pair \(((p, w); (\alpha, \beta; \gamma_0)\nu \in N)\), where \(p \in \mathbb{R}_+^m\) and \(w \geq 0\) such that:

(a) \(\forall \nu \in N, (\alpha\nu; \beta\nu; \gamma_0\nu) \in A\nu(p, w)\) (individual optimality);

(b) \(\alpha + \beta \leq \omega\) (social feasibility),
where \(\alpha \equiv \sum_{\nu \in N} \alpha\nu, \beta \equiv \sum_{\nu \in N} \beta\nu,\) and \(\omega \equiv \sum_{\nu \in N} \omega\nu;\)

(c) \(\beta_0 = \gamma_0\) (labour market equilibrium)
where \(\beta_0 \equiv \sum_{\nu \in N} \beta_0\nu, \gamma_0 \equiv \sum_{\nu \in N} \gamma_0\nu;\) and

(d) \(\hat{\alpha} + \hat{\beta} \geq Nb\) (reproducibility),
where \(\hat{\alpha} \equiv \sum_{\nu \in N} (\alpha\nu - \bar{\alpha}', \hat{\beta} \equiv \sum_{\nu \in N} (\bar{\beta} - \bar{\beta}').\) and \(\alpha_0 \equiv \sum_{\nu \in N} \alpha_0\nu.\)

\(^{13}\) A financial market might be introduced but it would not change the main results. For an interesting analysis, see chapter 3 in Roemer (1981).

\(^{14}\) As in Roemer (1980, 1981, 1982), \(MP'\) can be interpreted either as embodying an assumption of stationary price expectations, or as applying in a general expectations framework, at a stationary state, if one exists.
In other words, at a RS (a) all agents optimise; (b) aggregate capital is sufficient for production plans; (c) the labour market is in equilibrium; and (d) net output is sufficient for subsistence. The last condition (d) is equivalent to requiring that the vector of social endowments does not decrease component-wise, because (d) is equivalent to

\[ \omega - (\alpha + \beta + (\pi + \beta - N) b) \geq \omega, \]

which states that stocks at the beginning of next period should not be smaller than stocks at the beginning of the current period. Indeed, although the RS is defined as a temporary equilibrium in a static general equilibrium framework, it can be seen as a one-period feature of a stationary equilibrium in a dynamic general equilibrium framework.\(^{15}\) For the sake of brevity, in what follows, the notation \((p, w)\) is used to represent the RS \((\alpha^\nu; \beta^\nu; \omega^\nu)_{\nu \in N}\).

In order to avoid an excess of uninteresting technicalities, it is assumed, as in Roemer (1982), that agents who are able to reproduce themselves without working use just the amount of wealth strictly necessary to obtain their subsistence bundle \(b\): in a subsistence economy, wealthy agents have no reason to accumulate or to consume more than \(b\); hence, by stating that they do not “waste” their capital, assumption NBC is consistent with capitalist behaviour.

**Non Benevolent Capitalists (NBC):** If agent \(\nu\) has a solution to \(MP^\nu\) with \(\alpha^0_\nu + \gamma^0_\nu = 0\), then agent \(\nu\) chooses \((\alpha^\nu; \beta^\nu; \gamma^\nu_0)\) to minimise \(p\alpha^\nu + p\beta^\nu\).

It is now possible to prove some preliminary results. Lemma 1 proves that at a RS, the net revenue constraint binds for all agents.

**Lemma 1:** Assume NBC. Let \((p, w)\) be a RS for \(E = \langle N; (P, b); (\omega^\nu)_{\nu \in N}\rangle\). Then, \(p\alpha^\nu + p\beta^\nu - w\beta^\nu_0 + w\gamma^\nu_0 = pb\) for all \(\nu\).

The next Lemma proves that the wealth constraint binds for all agents who work at the solution to \(MP^\nu\).

**Lemma 2:** Let \((p, w)\) be a RS for \(E = \langle N; (P, b); (\omega^\nu)_{\nu \in N}\rangle\) such that \(p\alpha - w\alpha_0 > 0\). For any \(\nu \in N\), if \(\alpha^0_\nu + \gamma^0_\nu > 0\), then \(p\alpha^\nu + p\beta^\nu = p\omega^\nu\).

For any \((p, w)\) and any \(\alpha \in P\), define the profit rate \(\pi = \frac{p\alpha - w\alpha_0}{p\alpha}\), and let \(\pi^{\max} = \max_{\alpha \in P} \frac{p\alpha - w\alpha_0}{p\alpha}\). By optimality, it is immediate to prove that only

\(^{15}\)Roemer (1980) and Veneziani (2007) provide two alternative dynamic frameworks that generalise the one-period RS. See also Fleurbaey (1996) for a related dynamic approach.
production processes yielding the maximal profit rate will be activated. The next result proves an important property of the set of solutions of $MP^\nu$.

**Lemma 3:** Let $(p, w)$ be a price vector such that $\pi_{\text{max}} > 0$. If $(\alpha^\nu; \beta^\nu; \gamma^\nu_0)$ solves $MP^\nu$, then $(\alpha^\nu; \beta^\nu; \gamma^\nu_0)$ also solves $MP^\nu$ whenever $\alpha^\nu + \beta^\nu = \alpha^\nu_0 + \beta^\nu_0$ and $\gamma^\nu_0 - \beta^\nu_0 = \gamma^\nu_0 - \beta^\nu_0$.

The equilibrium price vector can be characterised.

**Proposition 1:** Let $(p, w)$ be a RS for $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$. Then (i) $p \geq 0$ with $pb > 0$; (ii) $\pi_{\text{max}} \geq 0$; (iii) $w > 0$.

In general convex cone economies it is not possible to prove that at a RS the price vector will be strictly positive. In fact, it is possible for some good to be produced as a joint product without being used as an input.

Proposition 2 derives aggregate net output in equilibrium.

**Proposition 2:** Let $(p, w)$ be a RS for $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$. If $p > 0$, then $\hat{\alpha} + \hat{\beta} = Nb$. Conversely, if $\hat{\alpha}_i + \hat{\beta}_i > Nb_i$ for some good $i$, then $p_i = 0$.

The next result derives the optimal amount of labour expended by every agent at a RS.

**Proposition 3:** Let $(p, w)$ be a RS for $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$. Then, $\alpha^\nu_0 + \gamma^\nu_0 = \max\{0, \frac{pb - \pi_{\text{max}} p\omega^\nu}{w}\}$ for all $\nu$.

By Proposition 3, it follows that agent $\nu$ will not work at the optimal solution if and only if $p\omega^\nu \geq \frac{pb}{\pi_{\text{max}}}$, which implies $\pi_{\text{max}} > 0$.

## 3 Exploitation and Class in Convex Subsistence Economies

The concept of exploitation in a general convex economy can now be introduced. First of all, if exploitation is conceived of as involving an unequal exchange of labour (or simply as labour exploitation), it is necessary to identify the normative benchmark, that is the normatively relevant amount of labour involved in the exchange, which is usually defined as the labour value
of an agent’s ‘labour power.’ Agent $\nu$ is said to be exploited (resp. an exploiter) if she performs more (resp. less) labour than the labour value of her ‘labour power.’ The amount of labour expended by the agent is unambiguously $\Lambda^\nu \equiv \alpha^\nu_0 + \gamma^\nu_0$, but there are various ways of defining the value of labour power, which is related to some reference bundle of commodities (e.g., that the agent does or can purchase). For any bundle $c \in \mathbb{R}^m_+$, the labour value of $c$ must be defined. Unlike in standard Leontief economies, the definition of the labour value of $c$ is not obvious, and various definitions have, in fact, been proposed (see Yoshihara, 2010, for a thorough discussion). Following are the most relevant ones discussed in the literature.

Definition 2 has been proposed by Morishima (1974). It suggests that the labour value of a given bundle of goods corresponds to the minimum amount of labour necessary to produce that bundle as net output. Given $c \in \mathbb{R}^m_+$, let $\phi(c) \equiv \{\alpha \in P | \hat{\alpha} \geq c\}$.

**Definition 2 [Morishima (1974)]:** Let $c \in \mathbb{R}^m_+$ be a given nonnegative bundle of commodities. Then, the labour value of $c$ is given by:

$$l.v.(c) = \min\{\alpha_0 \in \mathbb{R}_+ | \alpha \in \phi(c)\}.$$  

The next Definition has been proposed by Roemer (1982). It suggests that the labour value of a given bundle of goods corresponds to the minimum amount of labour necessary to produce that bundle as net output using a profit-rate maximising technique. Given a price vector $(p, w)$, let

$$\overline{P}(p, w) \equiv \left\{\alpha = (-\alpha_0, -\alpha, \alpha) \in P \mid \frac{p\alpha - (p\alpha + w\alpha_0)}{p\alpha} = \pi_{\text{max}}\right\}.$$  

**Definition 3 [Roemer (1982)]:** Let $c \in \mathbb{R}^m_+$ be a given nonnegative bundle of commodities. Then, the labour value of $c$ is given by

$$l.v.(c; p, w) = \min\{\alpha_0 \in \mathbb{R}_+ | \alpha \in \overline{P}(p, w) \cap \phi(c)\}.$$  

Instead of discussing the virtues and limitations of existing definitions of exploitation, and possibly introducing a new one, this paper adopts a novel approach and suggests starting from first principles, by defining the desirable properties that any definition of exploitation should satisfy. At the most general level, the UE notion of exploitation aims to describe a relational property of a given social structure by focusing on the distribution of labour
associated to a given resource allocation. In principle, there are many possible alternative definitions of exploitation, but one might argue that there should be some common structure characterising all forms of exploitation as the UE of labour, which characterises an admissible class of definitions. The next axiom represents a domain condition which precisely identifies the admissible domain of all forms of exploitation, thus specifying the relevant framework for the discussion of the properties of UE exploitation in the rest of the paper. Let \( B(p, b) \equiv \{ c \in \mathbb{R}_+^m \mid pc = pb \} \): \( B(p, b) \) is the set of bundles that cost exactly as much as the subsistence vector at prices \( p \). Then:

**Labour Exploitation in Subsistence Economies (LES):** Consider any economy \( E = (\mathcal{N}; (P, b); (\omega^\nu)_{\nu \in \mathcal{N}}) \in \mathcal{E} \). Let \( (p, w) \) be a RS for \( E \). Given any definition of exploitation, two subsets \( N^{ter} \subseteq \mathcal{N} \) and \( N^{ted} \subseteq \mathcal{N} \), \( N^{ter} \cap N^{ted} = \emptyset \), constitute the set of exploiters and the set of exploited agents if and only if there exist \( \overline{c}, \underline{c} \in B(p, b) \) such that there exist \( \alpha \overline{c} \in \phi(\overline{c}) \) with \( \alpha \overline{c} = c \) and \( \alpha \underline{c} \in \phi(\underline{c}) \) with \( \alpha \underline{c} = c \) such that \( \alpha_0 \overline{c} \geq \alpha_0 \underline{c} \) and for any \( \nu \in \mathcal{N} \),

\[
\begin{align*}
\nu &\in N^{ter} \iff \alpha_0 \overline{c} > \Lambda^\nu; \\
\nu &\in N^{ted} \iff \alpha_0 \underline{c} < \Lambda^\nu.
\end{align*}
\]

Axiom **LES** requires that, at any RS, the sets \( N^{ter} \) and \( N^{ted} \) are characterised by identifying two (possibly identical) reference commodity vectors \( \overline{c}, \underline{c} \in \mathbb{R}_+^m \). Both reference bundles \( \overline{c} \) and \( \underline{c} \) can be purchased by the consumer and they identify the value of labour power. Thus, if an agent \( \nu \in \mathcal{N} \) optimally works \( \Lambda^\nu \) to earn the income necessary to purchase the subsistence bundle \( b \), and \( \Lambda^\nu \) is less (resp. more) than the labour expended to produce \( \overline{c} \) (resp. \( \underline{c} \)), then she is regarded as expending less (resp. more) labour than the ‘value of labour power.’ According to **LES**, the set of such agents coincides with \( N^{ter} \) (resp. \( N^{ted} \)).

As the domain condition for the admissible class of exploitation-forms, **LES** captures the essential insights of the UE theory of exploitation in convex subsistence economies.\(^{16}\) Given any definition of exploitation, the sets \( N^{ter} \) and \( N^{ted} \) are identified: in the UE approach, the two sets, and the exploitation status of each agent \( \nu \), are determined by the difference between

\(^{16}\) It should be stressed that **LES** only applies to labour-based definitions of exploitation. It is not relevant, for example, for Roemer’s property-relations definition of exploitation that emphasises inequalities in ownership of productive assets. Similar versions of **LES** can be defined in different economies; see Yoshihara (2010).
the amount of labour that \( \nu \) ‘contributes’ to the economy, in some relevant sense, and the amount she ‘receives’, in some relevant sense. In convex subsistence economies, the former quantity is unambiguously given by the amount of labour performed, \( \Lambda^\nu \), whereas there are many possible UE views concerning the amount of labour that each agent receives, which incorporate different normative and positive concerns. As a domain condition, LES incorporates the main features of UE theory that are shared by all the main approaches proposed in the literature.

First, according to LES, the amount of labour that each agent receives depends on their equilibrium income, or more precisely, it is determined in equilibrium by some reference consumption vectors that agents can purchase. In the standard approach, the reference vector is unique and it corresponds to the bundle actually chosen by the agent. LES is much weaker in that allows for more than one reference vector and it only requires that the reference vectors be potentially affordable, even if they are not actually purchased.

Second, LES captures another key tenet of the UE theory of exploitation by stipulating that the amount of labour associated with each reference bundle - and thus potentially ‘received’ an agent - is related to the production conditions of the economy. More precisely, LES states that the reference bundles be technologically feasible as net output, and it defines their labour ‘content’ as the amount of labour necessary to produce them. Thus, the amount of labour ‘received’ by each agent - or, in the standard terminology, the value of labour power - is a function of the amount of social labour that is allocated to agents. It is worth noting that LES requires that the amount of labour associated with each reference bundle be uniquely determined with reference to production conditions, but it does not specify how such amount should be chosen, and there may be in principle many alternative ways of producing \( \tilde{\tau}, \tilde{\zeta} \), and of determining \( \alpha^\zeta_0, \alpha^\tau_0 \).

To be sure, one might argue that an even weaker version of LES should be imposed which allows for more than two reference bundles, and associated labour amounts, as well as for heterogeneous bundles across individuals. For example, one may argue that all affordable bundles should be considered. Although the objection may be important in principle, the restrictions on reference bundles are arguably mild and reasonable in the economies considered in this paper, and the axiom LES can be generalised. If individual exploitation status is monotonic in labour performed, ceteris paribus, then all the relevant information to determine exploitation status can be summarised in at most two reference bundles, and the associated amounts of labour. Sim-
ilarly, although LES might be generalised to include agent-specific reference bundles, this is redundant in the context of convex subsistence economies with agents of identical preference.

Finally, it is worth noting that the vectors \( \mathbf{c} \) and \( \xi \) in LES need not be uniquely fixed, and may be functions of \((p, w)\) and \(b\). Further, once \( \mathbf{c} \) and \( \xi \) are identified, the existence of \( \alpha_{\mathbf{c}} \) and \( \alpha_{\xi} \) is guaranteed by A3. The condition \( \alpha_{\mathbf{c}} \geq \alpha_{\xi} \) is necessary for \( N_{\text{ter}} \) and \( N_{\text{ted}} \) to be disjoint in every possible economy.

Various definitions of labour exploitation proposed in the literature are discussed below, which satisfy LES. These definitions may be partitioned into two main approaches, depending on how the value of labour power is defined. In what may be defined as the direct approach, the value of labour power is defined as the labour value of the bundles that agents actually consume, whatever the definition of labour value is. In the indirect approach, instead, the value of labour power is defined as the labour value of some reference bundle that agents can afford with their subsistence income, even though they do not necessarily purchase it.

Two definitions that follow the first approach are considered. The first one is an application to subsistence economies (in which every agent consumes the bundle \( b \)) of Morishima’s (1974) classical definition.

**Definition 4:** Let \((p, w)\) be a RS for \( E = (N; (P, b); (\omega^\nu)_{\nu \in N}) \). Agent \( \nu \) is exploited if and only if \( \Lambda^\nu > l.v.(b) \); she is an exploiter if and only if \( \Lambda^\nu < l.v.(b) \); and she is neither exploited nor an exploiter if and only if \( \Lambda^\nu = l.v.(b) \).

Definition 4 satisfies LES by choosing \( \mathbf{c} = \xi = b \), where \( \alpha_{\mathbf{c}} = \alpha_{\xi} \) is chosen to satisfy \( \alpha_{\mathbf{c}}^0 = \alpha_{\xi}^0 = l.v.(b) \). The second definition is a refinement of Morishima’s and is due to Roemer (1982).

**Definition 5:** Let \((p, w)\) be a RS for \( E = (N; (P, b); (\omega^\nu)_{\nu \in N}) \). Agent \( \nu \) is exploited if and only if \( \Lambda^\nu > l.v.(b; p, w) \); she is an exploiter if and only if \( \Lambda^\nu < l.v.(b; p, w) \); and she is neither exploited nor an exploiter if and only if \( \Lambda^\nu = l.v.(b; p, w) \).

Denote the production vector \( \alpha \in \overline{P}(p, w) \) with \( \alpha_0 = l.v.(c; p, w) \) by \( \alpha(c) \).

Definition 5 satisfies LES by choosing \( \mathbf{c} = \xi = \alpha(b) \) with \( \alpha_{\mathbf{c}} = \alpha_{\xi} = \alpha(b) \), or \( \mathbf{c} = \xi = b \) with \( \alpha_{\mathbf{c}} = \alpha_{\xi} = (\alpha_0(b), \alpha(b), \alpha(b) + b) \).
As an illustration of the second approach, two definitions are discussed. The first has been proposed by Yoshihara (2010) and Yoshihara and Veneziani (2011):

**Definition 6:** Let \((p, w)\) be a RS for \(E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle\). Agent \(\nu \in N\) is exploited if and only if \(\Lambda^\nu > \min_{c \in B(p,b)} l.v. (c; p, w)\); she is an exploiter if and only if \(\Lambda^\nu < \min_{c \in B(p,b)} l.v. (c; p, w)\), and she is neither exploited nor an exploiter if and only if \(\Lambda^\nu = \min_{c \in B(p,b)} l.v. (c; p, w)\).

Let \(c^* = \arg \min_{c \in B(p,b)} l.v. (c; p, w)\). Definition 6 satisfies LES by choosing \(\tau = \frac{c}{c^*} = \frac{\alpha}{\alpha^*} \) with \(\alpha^* = \alpha(c^*)\), or \(\tau = \frac{c}{c^*} = \alpha(c^*)\) with \(\alpha^* = \alpha(c^*)\).

The second example of the indirect approach is an extension of the so-called “New Interpretation,” originally developed by Dumenil and Foley [Dumenil (1980); Foley (1982)]\(^{17}\) to convex cone economies, which has been proposed by Yoshihara (2010). Let \((p, w)\) be a RS and let \(\alpha + \beta\) be the corresponding aggregate production point. Let \(\tau = \frac{c}{c^*} \) with \(\alpha^* = \alpha(c^*)\).

**Definition 7:** Let \((p, w)\) be a RS for \(E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle\). Agent \(\nu \in N\) is exploited if and only if \(\Lambda^\nu > \tau b (\alpha_0 + \beta_0)\); she is an exploiter if and only if \(\Lambda^\nu < \tau b (\alpha_0 + \beta_0)\), and she is neither exploited nor an exploiter if and only if \(\Lambda^\nu = \tau b (\alpha_0 + \beta_0)\).

Definition 7 satisfies LES by choosing \(\tau = \frac{c}{c^*} = \frac{\alpha}{\alpha^*} \) with \(\alpha^* = \alpha(c^*)\). By Proposition 2, it follows that \(\tau = \frac{1}{\tau} \).

The previous arguments provide strong support to the idea that LES does represent an appropriate domain condition in exploitation theory. LES is formally weak and it incorporates some arguably compelling and widely shared views on exploitation as the UE of labour. Thus, although it can be proved that the axiom is not trivial and not all definitions in the literature satisfy it,\(^{18}\) all of the major approaches do.\(^{19}\)

\(^{17}\)See also Lipietz (1982); for a recent survey see Mohun (2004).

\(^{18}\)For example, it can be proved that the subjectivist notion of labour exploitation based on workers’ preferences recently proposed by Matsuo (2008) does not satisfy LES, as shown in Yoshihara and Veneziani (2009). For a more thorough discussion, see Veneziani and Yoshihara (2011).

\(^{19}\)It is worth noting that based on Flaschel’s (1983) definition of additive labor values, it is possible to derive another formulation of labor exploitation that satisfies LES.
Let \( W^\nu \equiv p\omega^\nu \). Theorem 1 characterises the exploitation status of every agent, based on their initial wealth \( W^\nu \), for all definitions of exploitation satisfying axiom \textit{LES}.

**Theorem 1:** Let \((p, w)\) be a RS for \( E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) such that \( \pi^{\max} > 0 \). Then, for any formulation of labour exploitation satisfying \textit{LES}:

(i) agent \( \nu \) is an exploiter if and only if \( W^\nu > \frac{1}{\pi^{\max}} [pb - w\alpha_0] \);

(ii) agent \( \nu \) is exploited if and only if \( W^\nu < \frac{1}{\pi^{\max}} [pb - w\alpha_0] \); and

(iii) agent \( \nu \) is neither an exploiter nor exploited if and only if \( \frac{1}{\pi^{\max}} [pb - w\alpha_0] \leq W^\nu \leq \frac{1}{\pi^{\max}} [pb - w\alpha_0] \).

Theorem 1 is considerably more general than similar results derived by Roemer (1982), in that it applies to a whole class of definitions of exploitation, rather than a specific approach. Thus, for any definition of exploitation satisfying axiom \textit{LES}, Theorem 1 identifies the wealth cut-offs that partition the set of agents based on their exploitation status. Given the analysis in the previous section, one immediately notes that for each of the Definitions analysed there is a unique wealth cut-off. Thus, by Theorem 1, under Definition 4, exploitation status depends on whether wealth is higher than, lower than or equal to \( W^* = \frac{1}{\pi^{\max}} [pb - w l.v. (b)] \); similarly, under Definition 5, the wealth cut-off is \( W^* = \frac{1}{\pi^{\max}} [pb - w l.v. (b; p, w)] \); under Definition 6, the wealth cut-off is \( W^* = \frac{1}{\pi^{\max}} [pb - w \min_{c \in B(p, b)} l.v. (c; p, w)] \); finally, under Definition 7, the wealth cut-off is \( W^* = \frac{1}{\pi^{\max}} [pb - w (\alpha_0 + \gamma_0)] \).

Following Roemer (1982), classes can also be defined in this economy, based on the way in which agents relate to the means of production. At a RS of the subsistence economy, an individually optimal solution for an agent \( \nu \) consists of a vector \((\alpha^\nu; \beta^\nu; \gamma^\nu_0)\). Therefore, let \((a_1, a_2, a_3)\) be a vector where \( a_i = \{+, 0\}, i = 1, 2, \) and \( a_3 = \{+, 0\} \), where “+” means a non-zero vector in the appropriate place. Agent \( \nu \) is said to be a member of class \((a_1, a_2, a_3)\), if there is an individually optimal \((\alpha^\nu; \beta^\nu; \gamma^\nu_0)\) which has the form \((a_1, a_2, a_3)\). The notation \((+, +, 0)\) implies, for instance, that an agent works in her own ‘shop’ and hires others to work for her; \((+, 0, +)\) implies that an agent works both in her own ‘shop’ and for others, etc. Although there are seven possible classes in the subsistence economy, it can be proved that at a RS, the set of producers \( N \) can be partitioned into the following five, theoretically relevant classes.
\[ C^1 = \{ \nu \in N \mid A^\nu (p, w) \text{ has a solution of the form } (0, +, 0) \}, \]
\[ C^2 = \{ \nu \in N \mid A^\nu (p, w) \text{ has a solution of the form } (+, +, 0) \setminus (+, 0, 0) \}, \]
\[ C^3 = \{ \nu \in N \mid A^\nu (p, w) \text{ has a solution of the form } (+, 0, 0) \}, \]
\[ C^4 = \{ \nu \in N \mid A^\nu (p, w) \text{ has a solution of the form } (0, 0, +) \}, \]
\[ C^5 = \{ \nu \in N \mid A^\nu (p, w) \text{ has a solution of the form } (0, 0, +) \}. \]

The notation \((+, +, 0) \setminus (+, 0, 0)\) means that agent \(\nu\) is a member of class \((+, +, 0)\) but not of class \((+, 0, 0)\), and likewise for the other classes. As a first step in the analysis of classes, Lemma 4 proves that \((+, +, +)\) and \((0, +, +)\) are indeed redundant.

**Lemma 4:** Let \((p, w)\) be a given price vector. Let \(\nu\) be such that \(W^\nu > 0\) and \(\Lambda^\nu > 0\) at the solution of \(MP^\nu\). Let \(\nu\) belong to either \((+, +, +)\) or \((0, +, +)\). Then, exactly one of the following statements holds:

- if \(\gamma_0^\nu < \beta_0^\nu\) for all optimal \((\alpha^\nu; \beta^\nu; \gamma_0^\nu)\), then \(\nu \in (+, +, 0) \setminus (+, 0, 0)\);
- if \(\gamma_0^\nu = \beta_0^\nu\) for some optimal \((\alpha^\nu; \beta^\nu; \gamma_0^\nu)\), then \(\nu \in (+, 0, 0)\);
- if \(\gamma_0^\nu > \beta_0^\nu\) for all optimal \((\alpha^\nu; \beta^\nu; \gamma_0^\nu)\), then \(\nu \in (+, 0, +) \setminus (+, 0, 0)\).

Given Lemma 4, it is now possible to prove a generalisation of Roemer’s (1982) core result concerning the correspondence between class status and wealth. Theorem 2 proves that classes \(C^1\) to \(C^5\) are pairwise disjoint and exhaustive, and wealthier agents belong to the upper classes.

**Theorem 2:** Let \((p, w)\) be a RS for \(E = \langle N; (P, b); (\omega_\nu)_{\nu \in N} \rangle\) such that \(\pi^{\max} > 0\). Then:

(i) For any \(i \neq j\), \(C^i \cap C^j = \emptyset\) and \(\bigcup_{i=1}^5 C^i = N\);

(ii) For any \(\nu, \mu \in N\), if \(\nu \in C^i\) and \(\mu \in C^j\) with \(i < j\) then \(W^\nu > W^\mu\).

Theorems 1 and 2 identify two different partitions of the set of agents based, respectively, on their exploitation status and their relation to the means of production (more precisely, their position in the labour market). In both cases, an agent’s wealth is the main determinant of her position in the social structure, and therefore it is legitimate to ask whether class and exploitation status are related, as predicted in Marxian theory. This is the main object of analysis in the next section.
4 The Class-Exploitation Correspondence Principle

In classical Marxian theory, exploitation and classes are related: capitalists are exploiters and proletarians are exploited. On the one hand, this correspondence of class and exploitation status gives a specific normative relevance to the concept of class; on the other hand, it emphasises the importance of relations of production in the generation of exploitation. More formally, the Class-Exploitation Correspondence Principle (CECP) can be defined as follows.

Class-Exploitation Correspondence Principle (CECP) [Roemer (1982)]:

Given any economy $E \in \mathcal{E}$, at any RS with $\pi^{\text{max}} > 0$,

(A) every member of $C^1 \cup C^2$ is an exploiter.

(B) every member of $C^4 \cup C^5$ is exploited.

A fundamental contribution of Roemer’s theory is the proof of the CECP in the private-ownership economy, which is derived as a result of the analysis, rather than being assumed. Epistemologically, though, Roemer forcefully argues that the central relevance of the CECP in class and exploitation theory implies that it should be considered as a postulate, by requiring that any satisfactory definition of exploitation (and classes) satisfies the CECP. Consistently with this approach, this section analyses the CECP under different definitions of exploitation satisfying LES.

As a first step, it is useful to provide a characterisation of class status in general convex cone subsistence economies. Let $(p, w)$ be a RS such that $\pi^{\text{max}} > 0$. Let $\alpha_{\text{min}} \in \mathcal{F}(p, w)$ be such that $\frac{p \alpha_{\text{min}}}{\alpha_{\text{min}}} = \min_{\alpha \in \mathcal{F}(p, w)} \left[ \frac{p \alpha}{\alpha} \right]$ and $\pi^{\text{max}} p \alpha_{\text{min}} + w \alpha_{\text{min}} = pb$, and let $\alpha_{\text{max}} \in \mathcal{F}(p, w)$ be such that $\frac{p \alpha_{\text{max}}}{\alpha_{\text{max}}} = \max_{\alpha \in \mathcal{F}(p, w)} \left[ \frac{p \alpha}{\alpha} \right]$ and $\pi^{\text{max}} p \alpha_{\text{max}} + w \alpha_{\text{max}} = pb$. Note that $\alpha_{\text{min}}, \alpha_{\text{max}}$ are well-defined. The next proposition provides a precise characterisation of class status based on an agent’s wealth.

Proposition 4: Let $(p, w)$ be a RS for $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$ such that $\pi^{\text{max}} > 0$. Then,

(i) $\nu \in C^1 \iff W^\nu \geq \frac{pb}{\pi^{\text{max}}}$;

(ii) $\nu \in C^2 \iff p \alpha_{\text{max}} < W^\nu < \frac{pb}{\pi^{\text{max}}}$;

(iii) $\nu \in C^3 \iff p \alpha_{\text{min}} \leq W^\nu \leq p \alpha_{\text{max}}$; and
(iv) \( \nu \in C^4 \iff 0 < W^\nu < p_{\underline{\omega}_{\text{min}}}; \)

(v) \( \nu \in C^5 \iff W^\nu = 0. \)

In other words, Proposition 4 derives four wealth cut-off levels that partition the set of agents into five classes. The richest agents are big capitalists, who can reproduce themselves without working, and the poorest, property-less agents are proletarians, who can only sell their labour in order to survive. All other agents fall into the intermediate classes, based on their wealth. In principle, the wealth cut-offs identified in Proposition 4 may be different from those characterising exploitation status identified in Theorem 1, which depend on the actual definition of exploitation adopted.

The next theorem provides the general condition for the CECP to hold under any definition of exploitation satisfying LES. To see this, let \((p, w)\) be a RS with \(\pi_{\text{max}} > 0\), and let

\[
\Gamma(p, w) \equiv \{ \alpha \in P \mid p\tilde{\alpha} = pb \text{ and } p\tilde{\alpha} - w\alpha_0 \in [\pi_{\text{max}}p_{\underline{\omega}_{\text{min}}}, \pi_{\text{max}}p_{\underline{\omega}_{\text{max}}}] \}.
\]

Note that \(\Gamma(p, w)\) is the set of feasible production processes yielding a net output with the same monetary value as the subsistence vector and a profit revenue equal to the profit revenue under one of the maximal-profit-rate processes. Similarly, the set of net outputs associated with \(\Gamma(p, w)\) is defined as follows: \(\hat{\Gamma}(p, w) \equiv \{ \tilde{\alpha} \in \mathbb{R}_m^P \mid \alpha \in \Gamma(p, w) \}\). Then:

**Theorem 3:** Let \((p, w)\) be a RS for \(E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle\) such that \(\pi_{\text{max}} > 0\). For any definition of labour exploitation satisfying LES, the CECP holds if and only if \(\tilde{\tau}, \underline{\zeta} \in \hat{\Gamma}(p, w)\).

Again, Theorem 3 is significantly more general than the analogous results proved by Roemer (1982), in that it provides general necessary and sufficient conditions for the CECP to hold for an entire class of definitions of exploitation, rather than for a specific approach.

In other words, for any definition of labour exploitation satisfying LES, the CECP holds if and only if the profit revenue associated with the production of the reference bundles \(\tilde{\tau}\) and \(\underline{\zeta}\) is equal to the profit revenue under one of the maximal-profit-rate processes. This characterisation holds regardless of whether the amount of labour necessary to produce \(\tilde{\tau}, \underline{\zeta}\) is minimal or not, and regardless of whether the net output is actually produced at the RS. Note that this does not exclude the possibility that the CECP holds under a labour value formulation with no maximal-profit-rate production process.20

20 For a proof of this claim see Example A.1 in Appendix 2 below.
This property is due to the assumption of subsistence economies. In fact, in the case of accumulation economies, as Yoshihara (2010) shows, if the CECP holds under a labour value formulation, then this formulation should be associated with some maximal-profit-rate production process.  

Based on Theorem 3, it is possible to prove that the CECP holds under Definitions 5, 6, and 7.

**Corollary 1:** Let \((p, w)\) be a RS for \(E = (N; (P, b); (\omega^\nu)_{\nu \in N})\) such that \(\pi^{\text{max}} > 0\). Then the CECP holds under Definitions 5, 6, and 7.

Instead, Definition 4 does not preserve the CECP, because the commodity bundle used to define the value of labour power under Definition 4 need not be produced with the same profit revenue as that of the maximal-profit-rate processes. This is shown formally in the following proposition.

**Proposition 5:** Let exploitation be defined according to Definition 4. There is a convex cone, subsistence economy \(E \in \mathcal{E}\) in which there exists a RS with \(\pi^{\text{max}} > 0\), such that the CECP does not hold.

The results presented in this section are quite relevant for exploitation theory. Given the epistemological relevance of the CECP discussed above, Proposition 5 suggests that Morishima’s classical definition of exploitation is inadequate to capture the central intuitions of Marxian theory. Contrary to Roemer’s claims, however, Corollary 1 proves that, even in the general setting analysed in this paper, the CECP does hold for various definitions of exploitation presented in the literature. More generally, Theorem 3 suggests that, for all possible convex subsistence economies and all equilibria, the CECP holds for a whole class of notions of exploitation as UE of labour satisfying the arguably weak and reasonable axiom LES.

Finally, the following equivalence relation can be proved:

**Theorem 4:** Consider any definition of exploitation which satisfies LES.

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21 Note that Roemer (1982; Chapter 5; p.164) also stated basically the same claim, though he only discussed the case of exploiters.

22 Actually, Morishima’s definition has another arguably undesirable property. In fact, it is not difficult to construct an example of a subsistence economy in which all producers have the same endowment of capital goods, but they are all exploited in the sense of Definition 4, a rather counterintuitive result.

23 As shown by Yoshihara (2010), a similar result holds in accumulation economies.
Let \((p, w)\) be a RS for \(E = (N; (P, b); (\omega^\nu)_{\nu \in N})\) and \(\pi, \zeta \in \hat{\Gamma} (p, w)\) for this definition of exploitation. Then, the following statements are equivalent:

1. \(\pi^{\max} > 0\);
2. the \textbf{CECP} holds;
3. every producer in \(C^5\) is exploited.

Theorem 4 has two important implications for exploitation theory. Firstly, the equivalence between (1) and (3) represents a significant generalisation of the so-called “\textit{Fundamental Marxian Theorem}” (FMT) [Roemer (1980, 1981)], according to which the equilibrium maximal profit rate is positive if and only if every worker is exploited. The FMT is proved to hold in general convex cone economies with a complex class structure and for a whole set of definitions of exploitation satisfying axiom \textbf{LES}. This is important because, as for the \textbf{CECP}, although it is proved as a result, the epistemological status of the FMT in exploitation theory is usually that of an axiom, and alternative definitions of exploitation are often compared in terms of their ability to preserve it. Actually, one might discuss which axiom - the \textbf{CECP} or the FMT - is more relevant and whether imposing both significantly decreases the set of available definitions of exploitation. The second, and somewhat more surprising, implication of Theorem 4 is that for the whole class of definitions of exploitation satisfying \textbf{LES} with \(\pi, \zeta \in \hat{\Gamma} (p, w)\), the \textbf{CECP} and the FMT are equivalent, and therefore one need not choose among them: any definition preserving one, also preserves the other.

5 An Axiomatic Characterisation of Exploitation

The results presented in the previous sections are encouraging: there exist a set of definitions of exploitation as the unequal exchange of labour that satisfy the reasonable condition imposed by \textbf{LES}, and preserve the \textbf{CECP} (and the FMT) as a result, in the equilibrium of the private-ownership economy. Furthermore, both \textbf{LES} and the requirement that the \textbf{CECP} holds in general convex economies are not trivial, and some of the definitions proposed in the literature do not satisfy them. In terms of identifying and defending one definition of exploitation, however, the analysis developed so far is not conclusive as it cannot discriminate between a number of competing notions. In line with the novel axiomatic approach adopted in this paper, the main
purpose of this section is to propose and defend some additional theoretical conditions that any definition of labour exploitation should satisfy and then to provide a characterisation result.

In the rest of this paper, the following condition is assumed.

A4. Let \( \omega = \sum_{\nu \in N} \omega^\nu \). For all \( E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \in \mathcal{E} \):

\[
\omega \in \left\{ \alpha \in \mathbb{R}_+^m \mid \exists \alpha \in P \text{ s.t. } \alpha_l < N \text{ and } \alpha_i \geq Nb \right\}.
\]

Let a feasible allocation be an allocation \( (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N} \in (P \times P \times [0,1])^N \) such that Definition 1-(b), (c), and (d) are satisfied. It is immediate to see that a necessary condition for the existence of a feasible allocation in \( E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \in \mathcal{E} \), is that \( \omega \in \left\{ \alpha \in \mathbb{R}_+^m \mid \exists \alpha \in P \text{ s.t. } \alpha_l < N \text{ and } \alpha_i \geq Nb \right\} \). A4 is slightly stronger in that it rules out the possibility that the only \( \omega \in \left\{ \alpha \in \mathbb{R}_+^m \mid \exists \alpha \in P \text{ s.t. } \alpha_l < N \text{ and } \alpha_i \geq Nb \right\} \) is such that \( \alpha_l = N, \alpha = \omega \), and \( \alpha_i = Nb \). The latter type of economies can be seen as representing a stage of development characteristic of primitive communism and they are not particularly interesting for the purposes of this paper. Exploitation and classes should arguably be analysed in economies that are at a stage of development such that they can in principle produce a surplus.

Given LES as the domain axiom for the admissible class of exploitation-forms, let us introduce a system of axioms which would characterise appropriate forms of exploitation. Firstly, the following axiom captures an arguably essential feature of any theory of exploitation.

**Relational Exploitation (RE):** Consider a definition of exploitation satisfying LES. At any RS for \( \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \), the two subsets \( N^{ter} \) and \( N^{ted} \) are such that \( N^{ter} \neq \emptyset \) if and only if \( N^{ted} \neq \emptyset \).

From a formal viewpoint, axiom RE imposes a rather weak restriction on LES. From a theoretical viewpoint, it captures the crucial relational aspect inherent in exploitative relations, such that if an agent is exploited, she must be exploited by someone, and vice versa if an exploiter exists, she must be exploiting someone.

Secondly, the following axiom is also a natural requirement for any definition of exploitation.

**Independence (IND):** Consider a definition of exploitation satisfying LES. Given two economies \( E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) and \( E' = \langle N; (P, b); (\omega'^\nu)_{\nu \in N} \rangle \),
let \( ((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N}) \) and \( ((p, w), (\alpha'^\nu; \beta'^\nu; \gamma'^\nu)_{\nu \in N}) \) be their corresponding RSs such that \( \sum_{\nu \in N} \left( (\alpha^\nu_0, \alpha^\nu_0', \alpha^\nu_0) \right) + \left( (\beta^\nu_0, \beta^\nu_0', \beta^\nu_0) \right) = \sum_{\nu \in N} \left( (\alpha'^\nu_0, \alpha'^\nu_0', \alpha'^\nu_0) \right) + \left( (\beta'^\nu_0, \beta'^\nu_0', \beta'^\nu_0) \right) \).

Let \( (\tau, \xi) \) and \( (\tau', \xi') \) be the corresponding reference bundles for \( E \) and \( E' \), respectively. Then, \( \alpha^\tau_0 = \alpha'^\tau_0 \) and \( \alpha^\tau_0 = \alpha'^\tau_0' \).

That is, consider two economies whose initial endowments may differ. Suppose that a given price vector \( (p, w) \) is part of a RS for both economies such that the equilibrium aggregate labour inputs and the equilibrium aggregate net outputs are identical. Then, IND states that the amounts of labour associated with the exploitation reference bundles must be the same. In other words, if two economies share the same economic structure, except for arguably irrelevant factors (distribution of commodity endowments), and have the same equilibrium prices and aggregate equilibrium activities, then the upper and lower bounds for the value of labour power should be identical. It immediately shows that all of Definitions 4, 5, 6, and 7 satisfy this axiom.

The next axiom represents an invariant property in terms of change in the size of population. Given an economy \( E = (N; (P, b); (\omega^\nu)_{\nu \in N}) \in \mathcal{E} \), the economy \( E' = (N'; (P, b); (\omega'^\nu)_{\nu \in N'}) \in \mathcal{E} \) is the \( k \)-th replica economy of \( E \) if and only if for each \( \nu \in N \) with \( \omega^\nu \in \mathbb{R}^N_+ \) there are \( k \) agents \( (\nu^1, \ldots, \nu^k) \subset N' \) with \( (\omega^{\nu^1}, \ldots, \omega^{\nu^k}) = (\omega^\nu, \ldots, \omega^\nu) \). The \( k \)-th replica of \( E \) is denoted by \( kE = (kN; (P, b); (\omega^{\nu^i})_{\nu^i \in kN}) \). Moreover, for \( E, kE \in \mathcal{E} \), and for any RS \( ((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N}) \) of \( E \), consider \( ((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in kN}) \) such that for each \( \nu \in N \) with \( (\alpha^\nu; \beta^\nu; \gamma^\nu) \), there are \( k \) agents \( (\nu^1, \ldots, \nu^k) \subset kN \) such that \( ((\alpha^{\nu^1}; \beta^{\nu^1}; \gamma^{\nu^1}), \ldots, (\alpha^{\nu^k}; \beta^{\nu^k}; \gamma^{\nu^k})) = ((\alpha^\nu; \beta^\nu; \gamma^\nu), \ldots, (\alpha^\nu; \beta^\nu; \gamma^\nu)) \).

Then, this \( ((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in kN}) \) is a RS for \( kE \in \mathcal{E} \), and it is called the \( k \)-th replica RS. Then:

**Replication Invariance (RI):** Consider a definition of exploitation satisfying LES. Given any economy \( E = (N; (P, b); (\omega^\nu)_{\nu \in N}) \in \mathcal{E} \), let \( ((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N}) \) be a RS and let \( (\tau, \xi) \) be the corresponding reference bundles. Take any integer \( k \geq 1 \), and consider the \( k \)-th replica economy of \( E \), \( kE = (kN; (P, b); (\omega^{\nu^i})_{\nu^i \in kN}) \in \mathcal{E} \), and the \( k \)-th replica RS \( ((p, w), (\alpha^{\nu^i}; \beta^{\nu^i}; \gamma^{\nu^i})_{\nu^i \in kN}) \) with corresponding reference bundles \( (\tau', \xi') \). Then, \( \alpha^\tau_0 = \alpha'^\tau_0 \) and \( \alpha^\tau_0 = \alpha'^\tau_0' \).
Given that $P$ is a convex cone, $\text{RI}$ is a natural requirement, and all of the main definitions in the literature satisfy it.

Finally, the following axiom is of a standard technical condition. As a preliminary step, let us introduce the concept of efficiency that is relevant in convex subsistence economies.

**Definition 8:** An allocation $(\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in N} \in (P \times P \times [0, 1])^N$ is efficient for $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$ if it satisfies:

1. $\sum_{\nu \in N} \alpha^\nu + \sum_{\nu \in N} \beta^\nu \leq \sum_{\nu \in N} \omega^\nu$, $Nb \leq \sum_{\nu \in N} \tilde{\alpha}^\nu + \sum_{\nu \in N} \tilde{\beta}^\nu$, and $\sum_{\nu \in N} \gamma^\nu_0 = \sum_{\nu \in N} \gamma^\nu_0$; and there is no other allocation $(\alpha'^\nu; \beta'^\nu; \gamma'^\nu_0)_{\nu \in N}$ in $(P \times P \times [0, 1])^N$ such that the above (i) is satisfied and
2. $\alpha'^\nu + \gamma'^\nu \leq \alpha^\nu + \gamma^\nu_0$ for all $\nu \in N$, and $\alpha'^\nu + \gamma'^\nu < \alpha^\nu + \gamma^\nu_0$ for some $\nu^* \in N$.

In other words, if an allocation is efficient, it is impossible to reduce the labour time of some agent without increasing the amount of labour expended by someone else. It is also worth noting that if an allocation is efficient according to Definition 10, then even a benevolent social planner cannot improve on it by choosing $(\alpha'^\nu)_{\nu \in N} \in P^N$ such that $\sum_{\nu \in N} \alpha'^\nu \leq \sum_{\nu \in N} \omega^\nu$, $Nb \leq \sum_{\nu \in N} \tilde{\alpha}'^\nu$, and $\alpha'^\nu \leq \alpha^\nu + \gamma^\nu_0$ for all $\nu \in N$, and $\alpha'^\nu < \alpha^\nu + \gamma^\nu_0$ for some $\nu^* \in N$.

Then, the following is our next axiom:

**Continuity (CON):** Consider a definition of exploitation satisfying LES. Consider an economy $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$ and suppose that there exists an efficient RS $(\langle p, w \rangle, (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in N})$ for $E$, such that $\pi^\text{max} = 0$. Let $(\bar{\tau}, \bar{\omega})$ with $\alpha^\text{L}_{\gamma^\text{L}}$ be the reference bundles associated to the RS. Suppose that there exists an infinite sequence of economies $\{E^k\}_{k=1}^\infty$ such that $E^k = \langle N; (P, b); (\omega^{k\nu})_{\nu \in N} \rangle$ for each $k = 1, \ldots, \infty$ and $(\omega^{k\nu})_{\nu \in N} \rightarrow (\bar{\omega}^\nu)_{\nu \in N}$ as $k \rightarrow \infty$, and that there exists an infinite sequence of vectors $\{(p^k, w^k), (\alpha^{k\nu}; \beta^{k\nu}; \gamma^{k\nu}_0)_{\nu \in N}\}$, each of which constitutes an efficient RS for $E^k$ for each $k = 1, \ldots, \infty$, with its associated profit rate $\pi^{\text{max}k}$ and reference bundles $\{(\bar{\tau}^k, \bar{\omega}^k)\}$ with $\alpha^{\text{L}^k}_{\gamma^\text{L}^k}$. If $\pi^{\text{max}k}$, $\alpha^{k\nu; \beta^{k\nu}; \gamma^{k\nu}_0}_{\nu \in N}$, $\pi^{\text{max}}, \alpha^\nu; \beta^\nu; \gamma^\nu_0$, then $\pi^{\text{max}} = 0$.

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24 Thus, Definition 10 appropriately focuses on the socially relevant phenomenon, namely the minimisation of labour, but it gives no weight to the behavioural assumption encompassed in NBC, according to which big capitalists minimise capital outlay.
That is, consider a situation that in the economy $E$, there exists an efficient RS $(p, w, (\alpha^{\nu}; \beta^{\nu}; \gamma^{\nu})_{\nu \in N})$ for $E$, such that its associated maximal profit rate is zero, $\pi_{\text{max}} = 0$. Suppose that there exists a sequence of economies $\{E^k\}_{k=1}^{\infty}$ which converges to $E$, where each $E^k$ differs from $E$ only in the profile of commodity endowments. Suppose also correspondingly that there exists a sequence of efficient RSs and maximal profit rates $\left\{ \left( \pi^{\text{max}}_{\nu}, w^k, (\alpha^{k_{\nu}}; \beta^{k_{\nu}}; \gamma^{k_{\nu}})_{\nu \in N} \right) \right\}_{k=1}^{\infty}$ converges to $(\pi^{\text{max}}_{\nu}, w, (\alpha^{\nu}; \beta^{\nu}; \gamma^{\nu})_{\nu \in N})$.

Then, the corresponding sequence of the reference bundles $\left\{ \left( \alpha^0_{\nu}, \alpha^0_{\nu} \right) \right\}_{k=1}^{\infty}$ also converges to $(\alpha^0_{\nu}, \alpha^0_{\nu})$.

Note that Definition 5 satisfies CON. Also, Definition 6 satisfies it. Needless to say, Definition 4 and Definition 7 also satisfy it.

Before proving the main Theorem of this section, which provides a complete axiomatic characterisation of exploitation, some intermediate results are derived, which are also interesting in their own right. First of all, the next Lemma proves that all RS’s in which the wealth constraints bind, are efficient.

**Lemma 5:** Let $(p, w, (\alpha^{\nu}; \beta^{\nu}; \gamma^{\nu})_{\nu \in N})$ be a RS for $E = (N; (P, b); (\omega^{\nu})_{\nu \in N})$. If $\sum_{\nu \in N} p (\alpha^{\nu} + \beta^{\nu}) = \sum_{\nu \in N} p \omega^{\nu}$, then $(\alpha^{\nu}; \beta^{\nu}; \gamma^{\nu})_{\nu \in N}$ is efficient.

Lemma 5 suggests that the scarcity of capital is a sufficient condition for efficiency in equilibrium. Yet, it does not rule out the possibility of inefficient RS’s. In particular, if $(p, w, (\alpha^{\nu}; \beta^{\nu}; \gamma^{\nu})_{\nu \in N})$ is a RS with $\sum_{\nu \in N} p \left( \alpha^{\nu} + \beta^{\nu} \right) < \sum_{\nu \in N} p \omega^{\nu}$, then $(\alpha^{\nu}; \beta^{\nu}; \gamma^{\nu})_{\nu \in N}$ is not necessarily efficient. Formally, it is not possible to prove a Marxist version of the First Welfare Theorem because local nonsatiation may be violated in this economy for all those agents who do not work at the optimum. In fact, a necessary condition for a RS to be inefficient is the existence of big capitalists who can reproduce themselves without working, and who maintain capital scarcity “artificially” by minimising capital outlay, consistently with the assumption NBC. An example of an inefficient RS in a convex subsistence economy is discussed in detail in Appendix 3 below and the role of big capitalists is shown. This is an interesting case, as it suggests a relation between inefficiencies and extreme forms of injustice, viz. exploitative social relations. From a normative perspective, however, this is the least challenging case since there is no trade-off, at least locally, between an improvement in efficiency and a reduction of social injustices, namely exploitation. Even from a Marxian viewpoint, it may be
argued that exploitation theory should focus on the amount of labour time necessary to produce the subsistence bundle by adopting efficient production techniques.

Proposition 6 proves that the amount of labour performed is uniquely determined at all efficient equilibria of the subsistence economy.

**Proposition 6:** Let \( \mathbf{E} = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) and \( \mathbf{E}' = \langle N; (P, b); (\omega'^\nu)_{\nu \in N} \rangle \) be two economies such that \( \sum_{\nu \in N} \omega^\nu = \sum_{\nu \in N} \omega'^\nu \). Let \( ((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N}) \) and \( ((p', w'), (\alpha'^\nu; \beta'^\nu; \gamma'^\nu)_{\nu \in N}) \) be two efficient RSs for \( \mathbf{E} \) and \( \mathbf{E}' \), respectively. Then \( \alpha_0 + \gamma_0 = \alpha'_0 + \gamma'_0 \).

According to Proposition 6, for a given vector of aggregate productive assets, the amount of labour performed is independent of the distribution of endowments and it is equalised across all efficient RS’s. The next Lemma, instead, focuses on efficient and resource-egalitarian equilibria and proves that if an efficient RS exists, then the same price vector supports an egalitarian RS in which all individuals have the same vector of productive endowments and activate the same production process as self-employed agents.

**Lemma 6:** Let \( ((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N}) \) be a RS for \( \mathbf{E} = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle \) such that \( \sum_{\nu \in N} p (\alpha^\nu + \beta^\nu) = \sum_{\nu \in N} p \omega^\nu \). Then \( ((p, w), (\alpha^0; 0; 0)_{\nu \in N}) \) is a RS for \( \mathbf{E}' = \langle N; (P, b); (\omega'^\nu)_{\nu \in N} \rangle \), where \( \alpha'^\nu = \frac{\sum_{\nu \in N}(\alpha^\nu + \beta^\nu)}{N} \) and \( \omega'^\nu = \frac{w}{N} \) for all \( \nu \in N \).

Given the new axioms, \( \text{RI, RE, CON}, \) and \( \text{IND} \), the following characterisations can be derived.

**Theorem 5:** A definition of labour exploitation satisfies \( \text{LES, RI, RE, CON, and IND} \) if and only if for all \( E \in \mathcal{E} \) and every RS \( (p, w) \), \( \alpha^0_\mathbf{E} = \alpha^0 = \frac{\alpha^0_\mathbf{E} + \beta^0_\mathbf{E}}{N} \).

Theorem 5 can thus be taken as providing support to the extension of Dumenil-Foley’s “New Interpretation” (Definition 7) as the appropriate definition of exploitation, which is uniquely characterised from a rather small set.

\( ^{25} \)Such a characterisation by using \( \text{RE} \) still holds if \( \text{RE} \) is weakened so that its requirement is imposed only on any RS with a positive profit rate.
of arguably weak axioms. Actually, Theorem 5 has a rather striking implication: because, as shown above, virtually all of the most relevant definitions of exploitation proposed in the literature such as Definitions 4, 5, 6, and 7 satisfy axioms \text{LES}, \text{RI}, \text{CON}, and \text{IND}, it is the rather mild axiom \text{RE} - which requires the presence of exploiters, whenever some agent is exploited - that rules out all alternative definitions except Definition 7. By Theorem 5, not all definitions of exploitation satisfy these reasonable properties, and indeed our axiomatic analysis allows us to precisely identify the limits of the received definitions of exploitation: if Morishima’s celebrated definition is adopted, for example, it is not difficult to construct examples in which \text{RE} is violated and all agents are exploited.

Finally, from Theorems 3 and 5, the next result follows:

**Corollary 3:** Definition 7 is the sole formulation of labour exploitation satisfying \text{LES}, \text{RI}, \text{RE}, \text{CON}, and \text{IND}, and under which the CECP holds at every \text{RS} \((p, w)\) with \(\pi^{\text{max}} > 0\) for all \(E \in \mathcal{E}\).

Thus, the extension of Dumenil-Foley’s “New Interpretation” (Definition 7) is the sole formulation of exploitation which satisfies all axioms and which preserves an important property of exploitation theory, namely the CECP in general convex economies.

### 6 Concluding remarks

This paper provides a formal analysis of exploitation in convex cone subsistence economies with rational optimising agents. The distributive injustice associated with exploitative relations is rigorously explored and the relevance of a notion of exploitation that emphasises the unequal exchange of labour is defended. Firstly, a definition of exploitation is provided according to which exploitative relations involve an unequal distribution of (and control over) social labour, consistently with a normative approach that focuses on individual well-being freedom and the self-realisation of men. The main results of Roemer’s classical theory are generalised and it is proved that this definition - actually, a whole class of definitions of exploitation - preserves all the

\[26\] It is worth noting that under Definitions 4 and 5, there indeed exist some \(E \in \mathcal{E}\) and some \text{RS} \((p, w)\), such that \(l.v.(b) \neq \frac{\alpha_0 + \beta_0 N}{N}\) and \(l.v.(\alpha(b); p, w) \neq \frac{\alpha_0 + \beta_0 N}{N}\), so that Theorem 5 does capture a fundamental difference between alternative definitions.
classical Marxist insights on exploitation in a framework that is considerably more general than the standard Leontief, or von Neumann economies: every agent’s class and exploitation status emerges in the competitive equilibrium; there is a correspondence between an agent’s class and exploitation status; and the existence of exploitation is inherently linked to the existence of positive profits. Thus, many important properties of exploitation theory are shown to be considerably more robust than is generally thought.

Secondly, and perhaps more importantly, unlike in the previous literature, this paper develops a new, axiomatic approach to exploitation theory. The core intuitions behind the concept of exploitation are thus directly addressed, in order to derive the appropriate definition of exploitation from first principles, rather than from ad hoc attempts to eschew a number of technical or theoretical problems. The main positive and normative aspects of exploitation theory are explicitly formalised in a set of axioms that every labour-based definition of exploitation should arguably satisfy. The axioms seem rather weak and reasonable, but they uniquely characterise the definition of exploitation adopted in the paper, which focuses on the distribution of social labour and which, quite interestingly, is conceptually related to the so-called “New-Interpretation” (Dumenil, 1980; Foley, 1982).

Interestingly, though this main theorem is derived by presuming a specific class of convex subsistence economies, the characterisation of the NI-form as the unique proper definition of UE exploitation is robust, even if the domain of economies is extended from the class of subsistence economies to a more comprehensive class of convex economies including not only subsistence economies, but also economies in which agents accumulate and have possibly heterogeneous preferences. This is because the NI-form of exploitation satisfies all of the five axioms discussed herein even if the domain of economies is extended, though each of the five axioms should be properly reformulated to suit to the target domains of economies. Hence, in order to investigate the unique properness of the NI-form, the axiomatic analysis on the domain of subsistence economies is sufficient.

Finally, it may be worth noting briefly that the above mentioned axiomatic derivation of the proper definition of exploitation opens a window to various new problems which request further investigation of welfare implication of UE exploitation. For instance, according to the appropriate form of exploitation as UE-labour derived by Theorem 5, the existence of exploitative relations implies inequality in labour supplied to earn one unit of income, which might be represented by a distribution of labour per unit of income,
\[ \left( \frac{\Lambda^\nu}{p \nu} \right)_{\nu \in N}, \] where \( c^\nu \) is agent \( \nu \)'s consumption bundle purchased by her income. Then, it would be interesting to discuss the issue of how serious the current exploitative relations are by developing a measurement of exploitative relations. Indeed, though all of the previous main literature on exploitation theory have argued on the issue of what is the proper definition of exploitation to examine the existence of exploitation, none of them have discussed the issue of how to measure exploitative relations. However, given that the existence of exploitative relations implies inequality of opportunity for well-being as we have seen, this measurement issue is also of upmost importance as well as relevant, for instance, in the context of international comparison of nations in terms of access to well-being.

Taking this motivation for granted, a measure of exploitative relations would be given by a numerical function, which is to represent a social welfare ordering defined over alternative distributions of labour per unit of income, just as analogical to the framework of income inequality measurement. Since there are potentially so many social welfare orderings over such distributions, an interesting open question would be to axiomatically study what types of social welfare orderings are proper as the measures of exploitative relations. Notice, though such a framework looks similar to the framework of the income inequality measurement, the normative implication of UE exploitation is neither identical nor reduced to that of income inequality, which indicates that the background axiomatic systems to support the measures of exploitative relations may be qualitatively different from the axiomatic systems for income inequality measures. Moreover, the measures of exploitative relations may also be completely different from the so-called aggregate rate of surplus value (or, the aggregate rate of exploitation) traditionally discussed in Marxist economic theory. It would be an interesting future agenda to characterise the conceptual difference between the measure of exploitative relations and the aggregate rate of exploitation.

These small notes indicate that, starting from the axiomatic derivation of the proper definition of exploitation provided in this paper, we may cultivate plenty of interesting future research agendas for welfare theory of exploitation as unequal exchange of labour.
7 Appendix

7.1 Appendix 1: The existence of a RS

This appendix provides a complete characterisation of reproducible solutions. Let \( \alpha_0 (\omega) \equiv \max \{ \alpha_0 \mid \exists \alpha = (-\alpha_0, -\alpha) \in P \text{ s.t. } \alpha \leq \omega \} \). Given the production set \( P \) and a non-negative number \( \pi \geq 0 \), let

\[
\tilde{P}(\pi) \equiv \{ (-\alpha_0, \alpha - (1 + \pi) \alpha) \in \mathbb{R}_- \times \mathbb{R}^m \mid (-\alpha_0, -\alpha, \alpha) \in P \}.
\]

Then, let

\[
C \equiv \{ \omega \in \mathbb{R}^m_+ \mid \exists \alpha \in P \& \exists \pi \geq 0 : \alpha \leq \omega, \alpha \geq N b, \alpha \not\succ N b \& (-\alpha_0, N b - \pi \alpha) \in \partial \tilde{P}(\pi) \}.
\]

That is, \( \omega \in C \) implies that there exists a production vector that is feasible from \( \omega \) and produces \( N b \) as net output. Moreover, such a production point is \emph{weakly efficient} in the sense that \( (-\alpha_0, N b - \pi \alpha) \in \partial \tilde{P}(\pi) \) for some \( \pi \geq 0 \). Note that \( C \) is non-empty, since there exists \( \alpha_{N b} \in P \) such that \( (-\alpha_{N b}^0, N b - \pi \alpha_{N b}) \in \partial \tilde{P}(\pi) \) for some \( \pi = 0 \), where \( \alpha_{N b}^0 = \arg\min_{\alpha \in P : \alpha \geq N b} \alpha_0 \).

Then:

**Theorem A.1:** Let \( b \in \mathbb{R}^m_+ \), \( \omega \in \mathbb{R}^m_+ \). Under A1-A3, if there exists a reproducible solution (RS) for the economy \( E = (N; (P, b); (\omega^\nu)_{\nu \in N}) \), then \( \omega \in C \).

**Proof.** Let \( (p, w) \in \mathbb{R}^{m+1}_+ \) be a RS with its corresponding aggregate production vector \( \alpha + \beta \in P \setminus \{0\} \). First, \( \alpha + \beta \leq \omega \) and \( \alpha + \beta - N b \geq \alpha + \beta \) immediately follow from Definition 1(b)-(c). Next, Propositions 1(i) and 2 imply that \( \alpha + \beta - N b \not\succ \alpha + \beta \). Also, at the RS, \( \pi_{\text{max}} = \frac{p N b - w (\alpha_0 + \beta_0)}{p (\alpha_0 + \beta)} \geq 0 \) by Proposition 1(ii), and therefore \( p N b - \pi_{\text{max}} p (\alpha + \beta) - w (\alpha_0 + \beta_0) = 0 \geq p \alpha' - \pi_{\text{max}} p \alpha' - wo' \) for all \( (-\alpha_0', \alpha' - \pi_{\text{max}} \alpha') \in P (\pi_{\text{max}}) \). This implies that \( (-\alpha_0 + \beta_0, N b - \pi_{\text{max}} (\alpha + \beta)) \in \partial \tilde{P}(\pi_{\text{max}}) \) and \( (p, w) \) is a supporting price of it. Thus, \( \omega \in C \). \( \blacksquare \)

Given the production set \( P \), let

\[
\Pi^P \equiv \left\{ \pi \geq 0 \mid \exists (-\alpha_0, \alpha - \pi \alpha) \in \tilde{P}(\pi) : \min_{i=1,\ldots,m} \left( \frac{\alpha_i}{\alpha} \right) > \pi \right\}.
\]
Note that $\Pi^P$ has a positive $\pi$ by A2. Then, let
\[
C^* \equiv \left\{ \omega \in \mathbb{R}_+^n \mid \exists \alpha \in P \& \exists \pi \in \Pi^P : \alpha = \omega, \; \hat{\alpha} \geq N \beta, \; \hat{\alpha} \notin N \beta \& (-\alpha_0, N \beta - \pi \alpha_0) \in \partial \tilde{P}(\pi) \right\}.
\]

Theorem A.2: Let $b \in \mathbb{R}_+^m$, $\omega \in \mathbb{R}_+^m$ and $\alpha_0(\omega) \leq N$. Under $A1 \& A3$, if $\omega \in C^*$, then there exists $(\omega^\nu)_{\nu \in N} \in \mathbb{R}_+^{Nm}$ with $\sum_{\nu \in N} \omega^\nu = \omega$ such that there exists a reproducible solution $(\pi \nu, \omega^\nu, \nu \in N) \in \mathbb{R}_+^{m+1}$ with its associated aggregate production activity $\alpha^* \in P$ under the economy $E = \{N; (P, b); (\omega^\nu)_{\nu \in N}\}$.

Proof. By definition, if $\omega \in C^*$, there exists $\alpha^* = (\alpha^* - \alpha^0, \alpha^0, \alpha^0) \in P$ such that $\alpha^* = \omega, \bar{\alpha} - N \beta \geq \alpha^* \bar{\alpha} - N \beta \bar{\alpha}$, and there exists $\pi \in \Pi^P$ such that $(-\alpha^0, N \beta - \pi \alpha^0) \in \partial \tilde{P}(\pi)$. Therefore the aggregate production vector $\alpha^*$ satisfies Definition 1, parts (b) and (d). It must be shown that there is a price vector such that $\alpha^*$ emerges from individually optimal choices and that the labour market clears. Note that since it is assumed that $\alpha_0(\omega) \leq N$, it follows that $\alpha^* - \alpha^0 P \geq \alpha^* - N \beta$. Since $(-\alpha^0, N \beta - \pi \alpha^0) \in \partial \tilde{P}(\pi)$, by the supporting hyperplane theorem, there exists $(p, w) \in \mathbb{R}_+^{m+1}$ such that $pN \beta - \pi \alpha^* - w\alpha_0^0 \geq \hat{\alpha}^* - \pi \alpha^* - w\alpha_0^0$ for all $\alpha \in P$. Since $0 \in \partial \tilde{P}(\pi)$, $pN \beta - \pi \alpha^* - w\alpha_0^0 \geq 0$. Moreover, $pN \beta - \pi \alpha^* - w\alpha_0^0 = 0$ by the cone property of $\tilde{P}(\pi)$, which implies that $\alpha^* \in \tilde{P}(p, w)$. Finally, by $\pi \in \Pi^P$, there exists $(-\alpha^0, \hat{\alpha}^* - \pi \alpha^0) \in \tilde{P}(\pi)$ such that $\min_{i=1,...,m} \left( \frac{\alpha^0_i}{\alpha_i^0} \right) > 1$, which implies $p\hat{\alpha}^* - \pi \alpha^* > 0$, thus $p\hat{\alpha}^* - \pi \alpha^* - w\alpha_0^0 \leq 0$ implies $w > 0$ by A1.

Let $(\omega^\nu)_{\nu \in N} \in \mathbb{R}_+^{Nm}$ be such that $\sum_{\nu \in N} \omega^\nu = \omega$ and $\pi \alpha^\nu \in [p - w, p]$ for any $\nu \in N$. Note that there exists an assignment of $\omega^\nu$, since $\alpha_0^0 \leq N$. Thus, $pb - \pi \omega^\nu \geq 0$ for all $\nu \in N$. Then, let $\rho^\nu = \frac{\omega^\nu}{\pi}$ for all $\nu \in N$, and let $\beta^\nu = \frac{\rho^\nu}{\pi} \alpha^\nu$ for all $\nu \in N$. Then, by definition,
\[
pb^\nu - \pi \rho^\nu \geq 0
\]
for all $\nu \in N$. Thus, since $pb - \pi \omega^\nu \geq 0$ for all $\nu \in N$, $\gamma^\nu \geq 0$ for all $\nu \in N$. Since $\pi \omega^\nu \geq pb - w$ for all $\nu \in N$, $1 \geq \gamma^0$. Therefore by Lemma 3, $(0; \beta^\nu; \gamma^0)_{\nu \in N} \in \times_{\nu \in N} A^\nu(p, w)$. Furthermore
\[
\sum_{\nu \in N} \gamma^0 = \sum_{\nu \in N} \left( \frac{pb - \pi \beta^\nu}{w} \right) = (Nb - \pi \sum_{\nu \in N} \beta^\nu / w) = (Nb - \pi \alpha^0) / w.
\]
This implies $\sum_{\nu \in N} \gamma^\nu = \alpha^0 = \sum_{\nu \in N} \beta^0$, which completes the proof. ■
7.2 Appendix 2: Proofs

Proof of Lemma 1: Suppose, contrary to the statement, that \( p\alpha^\nu + \beta_0^\nu - w\beta_0^\nu + w\gamma_0^\nu > pb \) for some \( \nu \). If \( \alpha_0^\nu + \gamma_0^\nu > 0 \), then by the convex cone property of the production set \( P \), agent \( \nu \) can reduce either \( \gamma_0^\nu \) or \( \alpha_0^\nu \) without violating feasibility, which contradicts optimality. If \( \alpha_0^\nu + \gamma_0^\nu = 0 \), then by the convex cone property of the production set \( P \), agent \( \nu \) can reduce \( \beta_0^\nu \) without violating feasibility, which contradicts NBC. □

Proof of Lemma 2: Suppose, contrary to the statement, that \( p\alpha^\nu + \beta_0^\nu < p\omega^\nu \). Then, by increasing the investment of capital and hiring other agents, \( \nu \) can increase her profits and reduce her labour expended, while reaching subsistence, which contradicts optimality. □

Proof of Lemma 3: By the convexity of \( MP^\nu \), it follows that \( (\alpha^\nu; \beta^\nu) \) is technically feasible with \( \beta_0^\nu + \alpha_0^\nu = \beta_0^\nu + \alpha_0^\nu \). This implies that labour expenditure is the same in both production plans, since \( (\gamma_0^\nu + \alpha_0^\nu) - (\beta_0^\nu + \alpha_0^\nu) = (\gamma_0^\nu + \alpha_0^\nu) - (\beta_0^\nu + \alpha_0^\nu) \). By Lemma 1, noting that only processes that yield the maximal profit rate are operated, it is possible to write \( \pi^{\max}(p\alpha^\nu + p\beta^\nu) + \omega\alpha_0^\nu + \omega\gamma_0^\nu = pb \). But then, it is immediate to check that \( (\alpha^\nu; \beta^\nu; \gamma_0^\nu) \) yields the same amount of net revenue and capital outlay. □

Proof of Proposition 1: Part (i). Suppose, contrary to the statement, that \( p = 0 \), which implies \( pb = 0 \). Then, at the solution to \( MP^\nu \), it will be \( \alpha_0^\nu + \gamma_0^\nu = 0 \), all \( \nu \), and thus \( \alpha_0 + \gamma_0 = 0 \). However, by A1 \( \alpha_0 + \gamma_0 = 0 \) implies that \( \pi + \beta = 0 \), which contradicts part (d) of the Definition of RS.

Part (ii). Suppose, contrary to the statement, that \( \pi^{\max} < 0 \). By Lemma 1, for all \( \nu \), \( (p\alpha^\nu - \omega\alpha_0^\nu) + (p\beta^\nu - \omega\beta_0^\nu) + \omega\alpha_0^\nu + \omega\gamma_0^\nu = pb \) where \( (p\alpha^\nu - \omega\alpha_0^\nu) < 0 \) and \( (p\beta^\nu - \omega\beta_0^\nu) < 0 \). Hence, at the solution to \( MP^\nu \) it must be \( \alpha^\nu = \beta^\nu = 0 \), all \( \nu \), which contradicts part (d) of the Definition of RS.

Part (iii). Suppose, contrary to the statement, that \( w \leq 0 \). Then, at the solution to \( MP^\nu \), it will be \( \gamma_0^\nu = 0 \) for all \( \nu \). Further, \( \pi^{\max} \leq 0 \) can be ruled out, because by part (i), \( pb > 0 \). However, if \( \pi^{\max} > 0 \), then at the solution to \( MP^\nu \) it will be \( \beta_0^\nu > 0 \) for all \( \nu \) with \( p\omega^\nu > 0 \), contradicting part (c) of the Definition of RS. □

Proof of Proposition 2: By Lemma 1, the net revenue constraint of every agent holds as an equality. Summing over \( \nu \), one obtains \( p\hat{\alpha} + p\hat{\beta} - w\beta_0 + w\gamma_0 = \)
\( pNb \) and using part (c) of the Definition of RS, \( p\hat{\alpha} + p\hat{\beta} = pNb \). The result then follows by part (d) of the Definition of RS. ■

**Proof of Proposition 3:** By Lemma 1, \( p\hat{\alpha} + p\hat{\beta} - w\beta' - wo\gamma' + w(\gamma_0' + \alpha_0') = pb \) holds for all \( \nu \). Noting that only processes yielding the maximal profit rate will be used, the latter expression can be written as \( \pi^{\max}(p\hat{\alpha} + p\hat{\beta}) + wo\gamma' + w\gamma_0' = pb \) for all \( \nu \). The result then follows by Proposition 1(iii) and Lemma 2. ■

**Proof of Theorem 1:** Let \( \bar{\zeta}, \underline{\zeta} \in B(p, b) \) be the reference consumption bundles of a given definition satisfying LES. For this definition, \( \nu \in N^{ter} \) if and only if there is a \( \hat{\zeta} \in \phi(\bar{\zeta}) \) such that \( \hat{\zeta} = \bar{\zeta} \) and \( \alpha_0^0 > \alpha_0' + \gamma_0' \), \( \nu \in N^{ted} \) if and only if there is a \( \underline{\alpha} \in \phi(\underline{\zeta}) \) such that \( \underline{\alpha} = \underline{\zeta} \) and \( \alpha_0^{\underline{\alpha}} < \alpha_0' + \gamma_0' \), and \( \nu \in N \setminus (N^{ter} \cup N^{ted}) \) if and only if \( \alpha_0^{\bar{\zeta}} \leq \alpha_0' + \gamma_0' \leq \alpha_0^{\underline{\alpha}} \). Note that by A1, \( \alpha_0^{\bar{\zeta}} > 0 \) and \( \alpha_0^{\underline{\alpha}} > 0 \). Moreover, by Proposition 1(iii), it follows that \( wo\gamma' \geq wo\gamma' > 0 \).

Consider agent \( \nu \) with \( \alpha_0' + \gamma_0' = 0 \) at the solution to \( MP^\nu \). Since \( \alpha_0^0 > 0 \), such an agent is an exploiter by the above characterisation of \( N^{ter} \). By Proposition 3, it follows that \( W^\nu \geq \frac{pb}{\pi^{\max}} > \frac{1}{\pi^{\max}} [pb - wo\gamma'] \).

Next, consider any agent \( \nu \) with \( \alpha_0' + \gamma_0' > 0 \) at the solution to \( MP^\nu \). By Proposition 3, \( \alpha_0' + \gamma_0' = \frac{pb - \pi^{\max}W^\nu}{w} \) and therefore, by LES, \( \nu \) will be an exploiter if and only if \( \frac{pb - \pi^{\max}W^\nu}{w} < \alpha_0^{\bar{\zeta}} \), which holds if and only if \( W^\nu > \frac{1}{\pi^{\max}} [pb - wo\gamma'] \). The other two conditions follow in like manner. ■

**Proof of Lemma 4:** 1. First, note that by the convexity of \( MP^\nu \), it follows that if \( \gamma_0' < \beta_0' \) for some optimal \( (\alpha^\nu'; \beta^\nu'; \gamma_0') \) and \( \gamma_0'^\nu > \beta_0'^\nu \) for some other optimal \( (\alpha_0'^\nu; \beta_0'^\nu; \gamma_0'^\nu) \), then there is a solution to \( MP^\nu \) such that \( \gamma_0'^\nu = \beta_0'^\nu \). Therefore, the three cases in the statement are mutually exclusive and they decompose agents with \( W^\nu > 0 \) and \( \Lambda^\nu > 0 \) at the solution of \( MP^\nu \) into disjoint sets.

2. Suppose \( \gamma_0' < \beta_0' \) for all optimal \( (\alpha^\nu'; \beta^\nu'; \gamma_0') \). If \( \gamma_0' = 0 \), then clearly \( \nu \in (+, +, 0) \) because by assumption \( \Lambda^\nu > 0 \) and \( 0 = \gamma_0' < \beta_0' \). If \( \gamma_0' > 0 \), then construct \( (\alpha_0'^\nu; \beta_0'^\nu; \gamma_0'^\nu) \) such that \( \gamma_0'^\nu = 0 \), \( \beta_0'^\nu = \beta_0' - \gamma_0' \) and \( \alpha_0'^\nu = \alpha_0' + \gamma_0' \). By Lemma 3, \( (\alpha_0'^\nu; \beta_0'^\nu; \gamma_0'^\nu) \) is also optimal (it yields the same amount of net revenue, labour expenditure, and capital outlay). It is sufficient to show that \( \nu \notin (+, 0, 0) \). Suppose, contrary to the latter statement, that \( \nu \) has an optimal solution of the form \( (\alpha^\nu; 0; 0) \). As in Lemma 3, it is possible to show that \( \nu \) also has a solution \( (\alpha'^\nu; \beta'^\nu; \gamma'^\nu) \) such that \( \alpha'^\nu = 0 \), \( \beta'^\nu = \alpha^\nu \) and
\( \gamma_0^\nu = \alpha_0^\nu \). But this contradicts the assumption that all optimal solutions for \( \nu \) are such that \( \gamma_0^\nu < \beta_0^\nu \).

3. The other two cases are proved similarly. ■

Proof of Theorem 2: Part (i). First, \( \nu \in C^5 \) if and only if \( W^\nu = 0 \). Next, by Proposition 3, \( \nu \in C^1 \) if and only if \( W^\nu \geq \frac{pb}{\pi^{\max}} \). The result then follows from Lemma 4.

Part (ii). It is sufficient to prove that the wealth ordering in the statement is true for agents \( \nu, \mu \in N \) with \( 0 < W^\nu < \frac{p\phi}{\pi^{\max}} \) and \( 0 < W^\mu < \frac{p\phi}{\pi^{\max}} \).

By Lemma 4, it follows that \( \nu, \mu \in \bigcup_{i=2,3,4} C^i \). Suppose \( \nu \in C^4 \) and \( \mu \in C^3 \) but \( W^\nu > W^\mu \). (\( W^\nu = W^\mu \) is ruled out by the disjointedness of classes.) Since \( \mu \in C^3 \), by reversing the reasoning in Lemma 4, it is possible to show that \( \nu \) has an optimal solution such that \( \alpha^\nu = 0, \beta^\nu = \alpha^\mu \) and \( \gamma^\nu = \alpha_0^\nu \), with \( \beta_0^\nu = \gamma_0^\nu \). Next, by Lemma 3, consider \( \nu \)'s solutions of the form \((0; \beta^\nu; \gamma^\nu)\). Since working time is strictly decreasing in wealth, \( W^\nu > W^\mu \) implies \( \Lambda^\nu = \gamma_0^\nu < \Lambda^\mu = \gamma_0^\mu \). However, \( \beta_0^\mu = \gamma_0^\mu \) and \( \beta_0^\nu < \gamma_0^\nu \) implies \( \beta_0^\nu < \beta_0^\mu \) which is impossible given that \( W^\nu > W^\mu \) and thus agent \( \nu \) can hire more labour than \( \mu \) by investing in sectors with the maximal profit rate \( \pi^{\max} > 0 \). A similar argument proves that if \( \nu \in C^2 \) and \( \mu \in C^5 \) then \( W^\nu > W^\mu \). ■

Proof of Proposition 4: 1. By Lemma 4, Proposition 3, and Theorem 2(ii), it is sufficient to prove part (iii) of the statement.

2. Suppose \( p_{\underline{\alpha}}^{\min} \leq W^\nu \leq p_{\underline{\alpha}}^{\max} \). By Lemma 3, it is possible to consider \( \nu \)'s solutions of the form \((0; \beta^\nu; \gamma_0^\nu)\), without loss of generality. By optimality, at the solution to \( MP^\nu \), \( p_\beta^\nu \) \( = \) \( w\beta_0^\nu + w\gamma_0^\nu \) \( = \) \( pb \), or equivalently \( \pi^{\max} p_\beta^\nu + w\Lambda^\nu = pb \), with \( p_\beta = W^\nu \). But then, since \( p_{\underline{\alpha}}^{\min} \leq W^\nu \leq p_{\underline{\alpha}}^{\max}, \) by the convexity of \( P \), it follows that there exists some \( \alpha \in P \), such that \( \pi^{\max} p_{\underline{\alpha}} + w\alpha_0 = pb \), with \( p_{\underline{\alpha}} = W^\nu \). The latter equation implies that \( \alpha_0 = \Lambda^\nu \), and thus \( \Gamma^\nu \) has a solution of the form \((+; 0; 0)\).

3. Conversely, suppose that \( \nu \in C^3 \), so that \( \Gamma^\nu \) has a solution of the form \((+, 0, 0)\). This implies that there exists \( \alpha \in P \) such that \( \pi^{\max} p_{\underline{\alpha}} + w\alpha_0 = pb \), with \( p_{\underline{\alpha}} = W^\nu \), which implies \( p_{\underline{\alpha}}^{\min} \leq W^\nu \leq p_{\underline{\alpha}}^{\max} \). ■

Proof of Theorem 3: The result follows immediately from Theorem 1 and Proposition 4. ■

Proof of Corollary 1: 1. First, consider Definition 7. By Theorem 3, what needs to be shown is that \( p_{\underline{\alpha}}^{\min} \leq \frac{1}{\pi^{\max}} \left[ pb - w \left( \frac{(\alpha_0 + \beta_0)}{N} \right) \right] \leq p_{\underline{\alpha}}^{\max} \) holds.
Let $\bar{W} = \frac{1}{\pi_{\text{max}}} \left[ pb - \frac{w(\alpha_0 + \beta_0)}{N} \right]$. Then, $\pi_{\text{max}} \bar{W} + \frac{w(\alpha_0 + \beta_0)}{N} = pb$. By definition, $\bar{W} = \frac{p(\alpha + \beta)}{N}$ holds, since by Lemma 1, $p \left( \alpha + \beta \right) = Npb$ holds, and at a RS, only processes yielding the maximal profit rate are operated. Then, $\alpha_0 \max \leq \frac{\alpha_0 + \beta_0}{p(\alpha + \beta)} \leq \frac{\alpha_0 \min}{\bar{W} \min}$ by the optimality of production plans. Note that $\alpha_0^{\max} p(\alpha + \beta) = \frac{pb}{\alpha_0 \min} - w$, $\alpha_0 \max = \frac{pb}{\alpha_0 \min} - w$, and $\alpha_0^{\max} p(\alpha + \beta) = \frac{pb}{\alpha_0 \max} - w$, which imply that $\frac{pb}{\alpha_0 \min} - w \leq \frac{pb}{\alpha_0 \max} - w$. Thus, $\alpha_0 \max \leq \frac{\alpha_0 + \beta_0}{N} \leq \alpha_0 \min$ holds. Given that $\pi_{\text{max}} p(\alpha + \beta) = \frac{pb}{\alpha_0 \min} + w \alpha_0 \min = \pi_{\text{max}} \bar{W} + \frac{w(\alpha_0 + \beta_0)}{N}$, the last inequality implies that $p(\alpha + \beta) \max \leq \bar{W} \leq p(\alpha + \beta) \max$.

2. Consider next Definitions 5 and 6. As noted above, $\min_{c \in B(p,b)} l.v. \left( c; p, w \right) \leq l.v. \left( b; p, w \right) \leq t(\alpha_0 + \beta_0) = \frac{\alpha_0 + \gamma_0}{N}$. Note that $l.v. \left( b; p, w \right) \leq \frac{\alpha_0 + \gamma_0}{N}$ follows from $pb = \frac{p(\alpha + \beta)}{N}$. Hence, by Theorem 3 and step 1 of the proof, it follows that $p(\alpha + \beta) \min \leq \frac{1}{\pi_{\text{max}}} \left[ pb - w \frac{\alpha_0 + \gamma_0}{N} \right] \leq \frac{1}{\pi_{\text{max}}} \left[ pb - w \left( \min_{c \in B(p,b)} l.v. \left( c; p, w \right) \right) \right]$. Hence, what needs to be shown is $\frac{1}{\pi_{\text{max}}} \left[ pb - w \left( \min_{c \in B(p,b)} l.v. \left( c; p, w \right) \right) \right] \leq p(\alpha + \beta) \max$.

Let $c^* = \arg \min_{c \in B(p,b)} l.v. \left( c; p, w \right)$. By the definition of $l.v. \left( c^*; p, w \right)$, there exists $\alpha (c^*) \in \bar{P} \left( p, w \right)$ such that $\alpha_0 (c^*) = l.v. \left( c^*; p, w \right)$. In this case, $pc^* = p\alpha (c^*)$ holds. Let us show this. Suppose $pc^* < p\alpha (c^*)$ if $c^* \leq \alpha (c^*)$. Since $p\alpha (c^*) = \pi_{\text{max}} p(\alpha + \beta) (c^*) + w \alpha_0 (c^*)$, it follows that $pc^* < \pi_{\text{max}} p(\alpha + \beta) (c^*) + w \alpha_0 (c^*)$. This is a contradiction, since $t(\alpha (c^*) \in B(p,b)$ and $\alpha_0 (c^*) = \min_{c \in B(p,b)} l.v. \left( c; p, w \right)$. Thus, $pc^* = p\alpha (c^*)$. The last equation implies that $\alpha (c^*) \in \bar{P} \left( p, w \right)$ and $\alpha (c^*) \in \bar{P} \left( p, w \right)$. Note that $c^* \leq \alpha (c^*)$ by definition, $\alpha (c^*) \in P$ implies $(-\alpha_0 (c^*), -\alpha (c^*), c^* + \alpha (c^*)) \in P$ by A3. Since $pc^* = p\alpha (c^*)$, $\alpha (c^*) \in \bar{P} \left( p, w \right)$ implies that $(-\alpha_0 (c^*), -\alpha (c^*), c^* + \alpha (c^*)) \in \bar{P} \left( p, w \right)$ and $c^* \in \bar{P} \left( p, w \right)$. Thus, since $c^* \in \bar{P} \left( p, w \right)$ implies $pc^* - w \alpha_0 (c^*) \leq \pi_{\text{max}} p(\alpha + \beta) \max$, $\frac{1}{\pi_{\text{max}}} \left[ pb - w \left( \min_{c \in B(p,b)} l.v. \left( c; p, w \right) \right) \right] \leq p(\alpha + \beta) \max$ follows from $pc^* = pb$, which completes the proof.

**Proof of Proposition 5:** Consider the following von Neumann technology:

$$A = \begin{bmatrix} 2 & 2 & 10 \\ 2 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 & 16 \\ 10 & 10 & 0 \end{bmatrix}, \quad L = \begin{pmatrix} 1, 1, 4/3 \end{pmatrix}.$$
Given this data, the production possibility set \( P_{(A,B,L)} \) can be defined:

\[
P_{(A,B,L)} = \left\{ \alpha \in \mathbb{R}^- \times \mathbb{R}^2_+ \times \mathbb{R}^+ : \exists x \in \mathbb{R}^3_+: \alpha \leq (-Lx, -Ax, Bx) \right\}.
\]

Then, consider a convex cone subsistence economy defined by \( P_{(A,B,L)} \), \( b = (2,2) \), and \( \omega = (N, N) \). Let \((\omega')_{\nu \in N}\) be such that \( \omega' = (\delta', \delta'') \), where \( \delta' \leq 2 \), all \( \nu \in N \), and \( \omega^i = (2,2) \), some \( i \in N \).

Let \( e_j \in \mathbb{R}^3_+ \) and \( \alpha_j \), \( j = 1, 2, 3 \), be defined as in Example 1 above. Then, \( \alpha_1 \equiv (0, 8) \), \( \alpha_2 \equiv (4, 8) \), and \( \alpha_3 \equiv (6, 0) \). Moreover, \( \hat{P}(\alpha_0 = 1) = co \{ \alpha_1, \alpha_2, \frac{3}{4} \alpha_3, 0 \} \). In this economy, \( p = (1, 0) \) and \( w = 2 \) constitute a \( RS \) with \( \alpha' = 0 \), \( \beta'' = \frac{p_0}{2} \), and \( \gamma_0'' = 1 - \frac{\omega''}{2} \) for all \( \nu \in N \). The corresponding aggregate production is \( \alpha + \beta = \frac{N \alpha'}{2} \). In such a case, \( \pi(\alpha + \beta; p, w) \equiv \frac{\alpha^2}{p_0^2} - w \omega_0^2 = 1 \), whereas \( \pi(\alpha; p, w) \equiv \frac{\alpha^3}{p_0^3} - w \omega_0^3 < 0 \) and \( \pi(\alpha^3; p, w) \equiv \frac{\alpha^3}{p_0^3} - w \omega_0^3 = \frac{1}{2} = \pi(\frac{3}{4} \alpha^3; p, w) \). Thus, \( \hat{P}(p, w) = \{ \alpha \in P | \exists \lambda > 0 : \alpha = \lambda \alpha^2 \} \) and \( \beta'' \in \hat{P}(p, w) \) for all \( \nu \in N \).

Hence, \( p\alpha_{\min} = p\alpha_{\max} = p\frac{\alpha'}{2} = 1 \). In contrast, \( l.v. (b) = \frac{17}{36} = \frac{1}{2} \alpha_0 + \frac{1}{6} \alpha_0^3 \), so that \( \frac{1}{\max} |p\beta - w (l.v. (b))| = 2 - \frac{17}{18} = \frac{19}{18} > 1 \). Therefore, every agent \( \nu \in C^1 \cup C^2 \) with \( 1 < W'' < \frac{19}{18} \) is exploited, and the \( \text{CECP} \) does not hold in this economy if Definition 4 is adopted. \( \blacksquare \)

**Proof of Theorem 4:** From Theorem 3, there only remains to prove that \( (3) \Rightarrow (1) \). Suppose that \( \pi^{\text{max}} = 0 \). Then, every producer earns the income \( p\beta \) solely from the wage. Thus, to earn the same income, every producer supplies the same amount of labour which is equal to the labour input \( \frac{\alpha + \beta}{N} \) corresponding to the social production activity per capita. Note that if \( \pi^{\text{max}} = 0 \), then \( \pi(c, e) \in \hat{P}(p, w) \) implies \( w\omega_0^3 = w\omega_0^c = p\beta = w\frac{\alpha_0 + \beta}{N} \). Thus, \( \alpha_0^c = \frac{\alpha_0 + \beta}{N} = \alpha_0^c \), which implies that there is no exploiter nor exploited agent by \( \text{LES} \). \( \blacksquare \)

**Proof of Lemma 5:** Suppose that \( (\alpha'; \beta''; \gamma_0')_{\nu \in N} \) is not efficient. This implies that there exists another allocation \( (\alpha''; \beta''; \gamma''_0)_{\nu \in N} \) such that \( \sum_{\nu \in N} \alpha''_\nu \leq \sum_{\nu \in N} \alpha'_\nu \), \( N b \leq \sum_{\nu \in N} \alpha''_\nu + \sum_{\nu \in N} \beta''_\nu + \sum_{\nu \in N} \gamma''_0 = \sum_{\nu \in N} \gamma'_0 \), and \( \alpha''_\nu + \gamma''_0 \leq \alpha'_0 + \gamma'_0 \) for all \( \nu \in N \) and \( \alpha''_\nu + \gamma''_0 < \alpha'_\nu + \gamma'_0 \) for some
Then, premultiplying the first two inequalities by $p$, one obtains

$$
\sum_{\nu \in N} p\alpha^\nu + \sum_{\nu \in N} p\beta^\nu \leq \sum_{\nu \in N} p\omega^\nu ,
$$

$$
\sum_{\nu \in N} p\alpha^\nu + \sum_{\nu \in N} p\beta^\nu \geq pNb = \sum_{\nu \in N} p\alpha^\nu + \sum_{\nu \in N} p\beta^\nu ,
$$

$$
\sum_{\nu \in N} \alpha_0^\nu + \sum_{\nu \in N} \gamma_0^\nu < \sum_{\nu \in N} \alpha_0^\nu + \sum_{\nu \in N} \gamma_0^\nu .
$$

Let $\beta'' = \sum_{\nu \in N} \alpha_0^\nu + \sum_{\nu \in N} \beta_0^\nu$ and $\gamma_0'' = \sum_{\nu \in N} \alpha_0^\nu + \sum_{\nu \in N} \gamma_0^\nu$. By definition, $\beta_0'' = \gamma_0''$. Thus, since $p\beta'' = \sum_{\nu \in N} p\alpha^\nu + \sum_{\nu \in N} p\beta^\nu$, and $\gamma_0'' < \sum_{\nu \in N} \alpha_0^\nu + \sum_{\nu \in N} \gamma_0^\nu$, it follows that

$$
\frac{p\beta'' - w\beta_0''}{p\beta''} > \frac{pNb - w (\sum_{\nu \in N} \alpha_0^\nu + \sum_{\nu \in N} \gamma_0^\nu)}{p\beta''}.
$$

This implies $\sum_{\nu \in N} p\alpha^\nu + \sum_{\nu \in N} p\beta^\nu < p\beta''$, since $(\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N}$ is a RS and thus only processes yielding the maximum profit rate are operated. However, since $\sum_{\nu \in N} p\alpha^\nu + \sum_{\nu \in N} p\beta^\nu = \sum_{\nu \in N} p\omega^\nu$, this implies $p\beta'' > \sum_{\nu \in N} p\omega^\nu$, a contradiction.  }

**Proof of Proposition 6:** The result follows immediately noting that if $((p, w); (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N})$ is efficient then $\alpha_0 + \beta_0 \leq \sum_{\nu \in N} \alpha_0^\nu + \sum_{\nu \in N} \gamma_0^\nu$ for all feasible $(\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N}$ that satisfy Definition 10(i). Therefore $\alpha_0 + \beta_0 \leq \alpha_0^\nu + \gamma_0^\nu$. A similar argument holds for $((p', w'); (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N})$ proving $\alpha_0 + \beta_0 \leq \alpha_0^\nu + \gamma_0^\nu$, which establishes the desired result.

**Proof of Lemma 6:** By the construction of $(\alpha^\nu)_{\nu \in N}$, $((p, w); (\alpha^\nu; 0; 0)_{\nu \in N})$ satisfies Definition 1(b), (c), and (d). Therefore, only individual optimality needs to be proved. Since $\sum_{\nu \in N} p (\alpha^\nu + \beta^\nu) = \sum_{\nu \in N} p\alpha^\nu$, it follows that $\sum_{\nu \in N} p\alpha^\nu = \sum_{\nu \in N} p\omega^\nu$, so that $p\alpha^\nu = p\omega^\nu$ holds for all $\nu \in N$. By Lemma 1, $\sum_{\nu \in N} p (\alpha^\nu + \beta^\nu) = Npb$, and therefore $\sum_{\nu \in N} p\alpha^\nu = Npb$, so that $p\alpha^\nu = pb$ holds for all $\nu \in N$. Further, note that, by Lemma 5, the allocation $(\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N}$ is efficient, which implies that $\sum_{\nu \in N} (\alpha_0^\nu + \gamma_0^\nu)$ is the minimal amount of labour expenditure to produce $Nb$ as a social net output under the capital constraint $\sum_{\nu \in N} \omega^\nu$. This implies that $\alpha_0^\nu$ is the minimal labour expenditure to produce $b$ under the constraint $\omega^\nu$ for each $\nu \in N$. 39
Hence, \( \alpha_0^{\nu} \) is the solution of the problem \( MP^{\nu} \) given the price vector \((p, w)\) in the economy \( \langle N; (P, b); (\omega^{\nu})_{\nu \in N} \rangle \). Since \( \sum_{\nu \in N} \alpha^{\nu} = \sum_{\nu \in N} (\alpha^{\nu} + \beta^{\nu}) \) by definition, \((p, w)\) supports \( \sum_{\nu \in N} \alpha^{\nu} \) as a profit-rate maximizing production point. Therefore, \((p, w), (\alpha^{\nu}; 0; 0)_{\nu \in N}\) is a RS for the economy \( \langle N; (P, b); (\omega^{\nu})_{\nu \in N} \rangle \).

### 7.2.1 Proofs for Theorem 5

In this subsection of Appendix 2, proof of Theorem 5 is developed.

**Lemma I:** Given \( E = \langle N; (P, b); (\omega^{\nu})_{\nu \in N} \rangle \) and given a RS \( ((p, w), (\alpha^{\nu}; \beta^{\nu}; \gamma_0^{\nu})_{\nu \in N}) \),

\[
\left( \sum_{\nu \in N} (\alpha^{\nu} + \beta^{\nu}) \right), \sum_{\nu \in N} (\alpha_0^{\nu} + \beta_0^{\nu})
\]

is a solution of the following programme:

\[
(M) \min_{\alpha \in P} \pi^{\max} p_{\alpha} + w_{\alpha} \quad \text{s.t. } p_{\alpha} \geq pNb.
\]

**Proof.** Without loss of generality, consider \((p, w), \alpha^{p,w}\) where \(\alpha^{p,w} = \sum_{\nu \in N} (\alpha^{\nu} + \beta^{\nu})\). Suppose that \(\alpha^{p,w}\) is not a solution of \((M)\). Then, there is \(\alpha' \in P\) such that \(\pi^{\max} p_{\alpha'} + w_{\alpha'} < \pi^{\max} p_{\alpha^{p,w}} + w_{\alpha^{p,w}}\) and \(p_{\alpha'} \geq pNb\). Since \(\pi^{\max} p_{\alpha^{p,w}} + w_{\alpha^{p,w}} = pNb, pNb - (\pi^{\max} p_{\alpha'} + w_{\alpha'}) > 0\), however, the latter inequality implies \(\frac{\pi^{\max} p_{\alpha'} - w_{\alpha'}}{\pi^{\max} p_{\alpha^{p,w}} - w_{\alpha^{p,w}}} > 1\), a contradiction. \(\blacksquare\)

**Lemma II:** Given \( E = \langle N; (P, b); (\omega^{\nu})_{\nu \in N} \rangle \), let \((p, w), (\alpha^{\nu}; \beta^{\nu}; \gamma_0^{\nu})_{\nu \in N}\) be a RS with \(\pi^{\max} = 0\). Then, \(\sum_{\nu \in N} (\alpha_0^{\nu} + \beta_0^{\nu}) = l.v. (Nb)\).

**Proof.** Without loss of generality, consider \((p, w), \alpha^{p,w}\) where \(\alpha^{p,w} = \sum_{\nu \in N} (\alpha^{\nu} + \beta^{\nu})\). By Lemma I, \(w_{\alpha_0^{p,w}} = \min_{\alpha \in P; \alpha \geq Nb} w_{\alpha}\). Since, by definition, \(l.v. (Nb) = \min_{\alpha \in P; \alpha \geq Nb} \alpha\), and noting that at a RS \(w > 0\), it follows that \(\alpha_0^{p,w} \leq l.v. (Nb)\). However, by reproducibility and the definition of \(l.v. (Nb)\), it follows that \(\alpha_0^{p,w} < l.v. (Nb)\) is not possible at a RS. Hence, \(\alpha_0^{p,w} = l.v. (Nb)\). \(\blacksquare\)

Let \( P (Nb) \equiv \{ \alpha \in P \mid \alpha \in \arg \min_{\alpha \in P; \alpha \geq Nb} \alpha_0 \} \) and \( SP (Nb) \equiv \{ \alpha \in P (Nb) \mid \exists \alpha' \in P (Nb) : \alpha' \leq \alpha \} \). Take a small positive number \( \varepsilon > 0 \), and consider \( \alpha \in P \) such that \( \alpha' \geq Nb \) with \( \alpha' \neq Nb \) and \( \alpha_0 = l.v. (Nb) + \varepsilon \).

Let us denote the set of such elements by \( P (l.v. (Nb) + \varepsilon; Nb) \). Moreover, let us define:

\[
SP (l.v. (Nb) + \varepsilon; Nb) \equiv \{ \alpha \in P (l.v. (Nb) + \varepsilon; Nb) \mid \exists \alpha' \in P (l.v. (Nb) + \varepsilon; Nb) : \alpha' \leq \alpha \}.
\]
Lemma III: Let us take an $\alpha^N_b \in SP(Nb)$ and its corresponding $\alpha^\varepsilon \in SP \cap SP(l.v. (Nb) + \varepsilon; Nb)$ for any sufficiently small positive $\varepsilon > 0$. Consider an economy $E = (N; (P, b); (\omega^\nu)_{\nu \in N})$ with $\omega^\nu = \frac{\alpha^\varepsilon}{N}$ for each $\nu \in N$, there exists a price vector $(p, w) \in \mathbb{R}^{n+1}^+ \times \mathbb{R}^m$ such that $(p, w), (\alpha^\varepsilon^\nu; 0; 0)_{\nu \in N}$ is an efficient RS with $\pi_{\text{max}} > 0$.

Proof. Given that $\alpha^\varepsilon \in SP \cap SP(l.v. (Nb) + \varepsilon; Nb)$, $\alpha^\varepsilon$ is a solution of the following programme:

$$(M_1) \max_{\beta \in P} -\beta_0 \text{ s.t. } \beta \geq Nb; \alpha^\varepsilon \geq \beta; \beta_0 \in [0, N].$$

Moreover, without loss of generality, we may define an implicit convex function $\Gamma^P : \mathbb{R}^{2m+1}_+ \rightarrow \mathbb{R}$ as $\Gamma^P(\beta_0, \beta, \beta) \leq 0$ if and only if $\beta \in P$; and $\Gamma^P(\beta_0, \beta, \beta) = 0$ if and only if $\beta \in \partial P$. Then, by Theorem of Kuhn and Tucker (Berge, 1963, p. 227), there exists a vector $(\mu, r_\mu, \mu_0, \mu_{m+1}, \xi) \in \mathbb{R}^{2m+3}_+ \setminus \{0\}$ such that $\alpha^\varepsilon$ is a solution for the following programme:

$$\max_{(\beta_0, \beta, \beta) \in \mathbb{R}^{2m+1}_+} -\mu_0 \beta_0 + \sum_{i=1}^m \mu_i (\beta_i - Nb_i) + \sum_{i=1}^m r_\mu_i (\alpha^\varepsilon_i - \beta_i) + \mu_{m+1} (\alpha^\varepsilon_0 - \beta_0) - \xi \Gamma^P(\beta_0, \beta, \beta),$$

with $\sum_{i=1}^m \mu_i (\alpha^\varepsilon_i - Nb_i) = 0$. Then, $\xi > 0$ should hold, since otherwise there exists an infeasible production point $\beta \notin P$ which realises a higher value of the above Lagrangian than $\alpha^\varepsilon$. Then, $\mu_0 + \mu_{m+1} > 0$ should hold, since otherwise an inefficient production point $(-\beta_0, -\alpha^\varepsilon, \bar{\alpha}^\varepsilon)$ with $\beta_0 > \alpha^\varepsilon_0$ realises a higher value of the above Lagrangian than $\alpha^\varepsilon$ by $-\xi \Gamma^P(\beta_0, \alpha^\varepsilon, \bar{\alpha}^\varepsilon) > 0$. Finally, for any commodity $i$ with $\alpha^\varepsilon_i = Nb_i$, $\mu_i > 0$ holds, since otherwise $(-\alpha^\varepsilon_0, -\alpha^\varepsilon, (\beta_i, \bar{\alpha}^\varepsilon))$ with $\beta_i < Nb_i$ realises a higher value of the above Lagrangian than $\alpha^\varepsilon$ by $-\xi \Gamma^P(\alpha^\varepsilon_0, \alpha^\varepsilon, (\beta_i, \bar{\alpha}^\varepsilon)) > 0$. Note that by the definition of $\alpha^\varepsilon$, there exists at least one commodity $i$ with $\alpha^\varepsilon_i = Nb_i$.

Let $p \equiv \sum_\mu p_\mu$ and $w \equiv \sum_\mu w_\mu$. Then, for any $\beta \in \partial P$, $p\beta - rp\beta - w\beta_0 \leq pNb - r\alpha^\varepsilon - w\alpha^\varepsilon_0 = p\alpha^\varepsilon - r\alpha^\varepsilon - w\alpha^\varepsilon_0$. By the cone property, $pNb - r\alpha^\varepsilon - w\alpha^\varepsilon_0 = 0$ holds. Then, $\alpha^\varepsilon \in T(p, w)$ with its associated maximal profit rate $\pi_{\text{max}} \equiv r$. This implies that $(p, w), (\alpha^\varepsilon^\nu; 0; 0)_{\nu \in N}$ constitutes
an efficient RS with $\pi^{\text{max}} > 0$ for the economy $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$ with $\omega^\nu = \frac{\alpha^\nu}{N}$ for each $\nu \in N$. Note that $\pi^{\text{max}} > 0$ follows from Lemma II and $\alpha_0^\nu > l. v. (N)$. ■

By Lemma III, without loss of generality, let $\alpha^\varepsilon \equiv (-\alpha_0^\nu, -\alpha_0^\nu + Nb)$ in the following discussion.

**Lemma IV:** Consider an economy $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$ with $\omega^\nu = \omega^{\nu'}$ for each $\nu, \nu' \in N$ and suppose that there exists an efficient RS $\langle (p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N} \rangle$ for $E$, such that $\pi^{\text{max}} = 0$ with $\sum_{\nu \in N} (\alpha^\nu + \beta^\nu) = \sum_{\nu \in N} \omega^\nu$. Then, there exists an infinite sequence of economies $\{E^k\}_{k=1}^{\infty}$ such that $E^k = \langle N; (P, b); (\omega^{k\nu})_{\nu \in N} \rangle$ for each $k = 1, \ldots, \infty$ and $(\omega^{k\nu})_{\nu \in N} \to (\omega^\nu)_{\nu \in N}$ as $k \to \infty$, and that there exists an infinite sequence of vectors $\left( (p^k, w^k), (\alpha^{k\nu}; \beta^{k\nu}; \gamma_0^{k\nu})_{\nu \in N} \right)$ associated with $\pi^{\text{max}} > 0$, each of which constitutes an efficient RS for $E^k$ for each $k = 1, \ldots, \infty$, such that $\left( \pi^{\text{max}} w^k, (\alpha^{k\nu}; \beta^{k\nu}; \gamma_0^{k\nu})_{\nu \in N} \right) \to (\pi^{\text{max}}, w, (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N})$ as $k \to \infty$.

**Proof.** Consider the economy $E = \langle N; (P, b); (\omega^\nu)_{\nu \in N} \rangle$ and suppose that there exists an efficient RS $\langle (p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N} \rangle$ for $E$, such that $\pi^{\text{max}} = 0$. Consider $\langle (p, w), \alpha^{p,w} \rangle$ where $\alpha^{p,w} = \sum_{\nu \in N} (\alpha^\nu + \beta^\nu)$ and $(p, w) \in \Delta^{m+1} = \{ (p, w) \in \mathbb{R}^m \times \mathbb{R}_+ \mid \sum_{i=1}^m p_i + w = 1 \}$, without loss of generality. If there is more than one $\alpha \in \phi(N)$ that can be part of a RS at prices $(p, w)$, then choose $\alpha \in \phi(N) \cap \overline{P}(p, w)$ that minimises $p\alpha$ as $\alpha^{p,w}$. [This is well defined since $\phi(N)$ is closed and bounded.] Also, without loss of generality, $\omega^\nu = \frac{\alpha^{p,w}}{N}$ for each $\nu \in N$.

Suppose that $w$ is the unique wage rate in $\Delta^{m+1}$ which supports $\alpha^{p,w}$. In other words, if $\langle p', w' \rangle \in \Delta^{m+1}$ is another price vector which supports $\alpha^{p,w}$ as a profit rate maximiser, then $w' = w$. This implies that the associated maximal profit rate of $\langle p', w' \rangle$ is zero. Take a sufficiently small neighborhood $\mathcal{N}(\alpha^{p,w})$ of $\alpha^{p,w}$ in $SP$. Take also a sequence of appropriately small positive number $\{\varepsilon^k\}_{k=1}^{\infty}$ such that $\varepsilon^k \to 0$ as $k \to \infty$. Then, there exists a sequence $\{\alpha^{\varepsilon^k}\}_{k=1}^{\infty} \subseteq \mathcal{N}(\alpha^{p,w}) \cap SP$, where $\alpha^{\varepsilon^k}$ is defined following the proof of Lemma III if $(\mathcal{N}(\alpha^{p,w}) \cap SP) \setminus \{t\alpha^{p,w} \mid t > 0\} \neq \emptyset$; otherwise, $\alpha^{\varepsilon^k} \equiv \alpha^{p,w}$ for any $k$. Note that $\alpha^{p,w} \in SP(N)$ and $(\mathcal{N}(\alpha^{p,w}) \cap SP) \setminus \{t\alpha^{p,w} \mid t > 0\} \neq \emptyset$ imply $SP \cap SP(l. v. (N) + \varepsilon^k; N) \neq \emptyset$, thus it is possible to take $\alpha^{\varepsilon^k} \equiv \arg \min_{\alpha \in SP \cap SP(l. v. (N) + \varepsilon^k; N)} \| \alpha - \alpha^{p,w} \|$. Then, $\alpha^{\varepsilon^k} \to \alpha^{p,w}$ as $k \to \infty$.  

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Case 1: \((N(\alpha^{p,w}) \cap SP) \setminus \{t\alpha^{p,w} | t > 0\} = \emptyset\). In this case, there exists another price \((p', w') \in \Delta^{m+1}\) such that \(\alpha^{p,w} \in \mathcal{F}(p', w')\) and its associated profit rate is positive. For any \(\lambda \in [0, 1]\), let \((p^\lambda, w^\lambda) \equiv (\lambda p + (1 - \lambda)p', \lambda w + (1 - \lambda)w') \in \Delta^{m+1}\). Then, it is easy to see that \(\alpha^{p,w} \in \mathcal{F}(p^\lambda, w^\lambda)\), so that \(\left(\left\langle (p^\lambda, w^\lambda), (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}}\right\rangle \right)\infty_{k'=1}\) is a RS for \(E\). Thus, there exists a sequence \(\left\{(\pi^{max}\lambda^{k'}, w^{\lambda^{k'}}, (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}})\right\}\infty_{k'=1}\) such that \(\lambda^{k'} = 0\) and \((\pi^{max}\lambda^{k'}, w^{\lambda^{k'}}) = (p', w')\) if \(k' = 1\); \((\pi^{max}\lambda^{k'}, w^{\lambda^{k'}}) = (\lambda^{k'}p + (1 - \lambda^{k'})p', \lambda^{k'}w + (1 - \lambda^{k'})w')\) for any \(k'\); and \(\lambda^{k'} \rightarrow 1\) as \(k' \rightarrow \infty\). Thus, \((\pi^{max}\lambda^{k'}, w^{\lambda^{k'}}, (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}}) \rightarrow (\pi^{max}, w, (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}})\) where \(\pi^{max}\lambda^{k'} > 0\) whenever \(\lambda^{k'} < 1\).

Case 2: \((N(\alpha^{p,w}) \cap SP) \setminus \{t\alpha^{p,w} | t > 0\} \neq \emptyset\). In this case, by Lemma III, we can construct a sequence of economies \(\{E_k\}_{k=1}^{\infty}\) where \(\omega^{kv} = \frac{\lambda^v}{N}\) for each \(\nu \in \mathbb{N}\), and its corresponding sequence of RSs \(\left\{(\pi^{max}\lambda^{k'}, w^{\lambda^{k'}}, (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}})\right\}\infty_{k=1}\) with \((p^k, w^k) \in \Delta^{m+1}\) and \(\pi^{max} > 0\). Clearly, \((\alpha^{k\nu}; \beta^{k\nu}; \gamma^\nu_0)_{\nu \in \mathbb{N}} \rightarrow (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}}\) as \(k \rightarrow \infty\). Then, by Lemma I and Berge’s Maximum Theorem, \((p^k, w^k) \rightarrow (p^*, w) \in \Delta^{m+1}\) as \(k \rightarrow \infty\), such that \((\pi^{max}, w, (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}})\) is a RS associated with zero profit rate for \(E\). Thus, \(w^k \rightarrow w\) as \(k \rightarrow \infty\).

Then, again by the Berge Maximum Theorem, there exists a continuous function \(\Psi : \left\{(w^k(\alpha^{k\nu}; \beta^{k\nu}; \gamma^\nu_0)_{\nu \in \mathbb{N}})\infty_{k=1} \right\} \rightarrow \mathbb{R}_+\) such that \(\Psi(w^k(\alpha^{k\nu}; \beta^{k\nu}; \gamma^\nu_0)_{\nu \in \mathbb{N}}) = \pi^{max}k\) for each \(k\). This implies that there exists an infinite sequence of vectors \((p^k, w^k(\alpha^{k\nu}; \beta^{k\nu}; \gamma^\nu_0)_{\nu \in \mathbb{N}})\) associated with \(\pi^{max} > 0\), each of which constitutes an efficient RS for \(E_k\) for each \(k = 1, \ldots, \infty\), such that \((\pi^{max}k, w^k(\alpha^{k\nu}; \beta^{k\nu}; \gamma^\nu_0)_{\nu \in \mathbb{N}}) \rightarrow (\pi^{max}, w, (\alpha^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}})\) as \(k \rightarrow \infty\).

Let there exist another price \((p', w') \in \Delta^{m+1}\) such that \(\alpha^{p,w} \in \mathcal{F}(p', w')\) and its associated profit rate is positive. Then, the argument developed in Case 1 can be applied so that the desired result is obtained.

Proof of Theorem 5:

\((\Leftarrow)\): It is easy to see that Definition 7 meets \(\text{IND}, \text{RI}, \text{and CON}\). Let us see that Definition 7 meets \(\text{RE}\). Let \((\pi^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}}\) be an efficient RS for the economy \((N; (P, b); (\omega^\nu)_{\nu \in \mathbb{N}})\). Then, if \(\frac{\alpha^\nu + \beta^\nu}{N} = 1\), there is no exploiter not exploited agent under Definition 7. If \(\frac{\alpha^\nu + \beta^\nu}{N} < 1\), then somebody is exploited if and only if somebody else is an exploiter according to Definition 7. Next, let \((\pi^\nu; \beta^\nu; \gamma^\nu_0)_{\nu \in \mathbb{N}}\) be an inefficient RS. Thus,
by Lemma 5 and Lemma 2, there exists $\nu' \in N$ such that $\alpha_0^\nu' + \gamma_0^\nu' = 0$. This implies $\frac{\alpha_0 + \beta_0}{N} < 1$. Again, somebody is exploited if and only if somebody else is an exploiter according to Definition 7. In sum, Definition 7 meets RE.

$(\Rightarrow)$: Consider any definition of labour exploitation satisfying LES, CON, RE, RI, and IND. Consider an economy $E = \langle N; (P, b); (\nu^\nu)_{\nu \in N} \rangle$ with a RS $\langle (p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N} \rangle$.

Case 1: $\pi_{\text{max}} > 0$ and $\alpha + \beta = \sum_{\nu \in N} \omega^\nu = \omega$. By Lemma 5, $(\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in N}$ is efficient, and $\frac{\alpha + \beta}{N} < 1$ by A4. Suppose the condition $\alpha_0^\nu = \alpha_0 = \frac{\alpha_0 + \beta_0}{N}$ does not hold.

1. Suppose $\alpha_0^\nu < \frac{\alpha_0 + \beta_0}{N}$. Consider another economy $E' = \langle N; (P, b); (\nu^\nu)_{\nu \in N} \rangle$ with $\omega^\nu = \frac{\omega}{N}$ for all $\nu \in N$, and an allocation $(\alpha^\nu'; \beta^\nu'; \gamma_0^\nu)_{\nu \in N} \in (P \times P \times [0, 1])^N$ such that $\alpha^\nu' = \frac{\alpha + \beta}{N}$, $\beta^\nu' = \beta$, and $\gamma_0^\nu = 0$ for all $\nu \in N$. By construction, $\langle (p, w), (\alpha^\nu'; \beta^\nu'; \gamma_0^\nu)_{\nu \in N} \rangle$ constitutes a RS for $E'$ with $\pi_{\text{max}} > 0$ and $\alpha' + \beta' = \alpha + \beta$. Then, by IND, its corresponding reference bundles $(\pi', \omega')$ meet $\alpha_0' = \alpha_0$ and $\alpha_0^\nu = \alpha_0$. Because $\Lambda' = \frac{\alpha_0 + \beta_0}{N}$ for all $\nu \in N$, at $\langle (p, w), (\alpha^\nu'; \beta^\nu'; \gamma_0^\nu)_{\nu \in N} \rangle$, then $\alpha_0^\nu < \frac{\alpha_0 + \beta_0}{N}$ implies that every agent is exploited, which contradicts RE.

2. Suppose $\alpha_0' > \frac{\alpha_0 + \beta_0}{N}$. Consider the same economy and the same allocation as in step 1 above. Because $\alpha_0' > \alpha_0$, every agent is an exploiter at the RS $\langle (p, w), (\alpha^\nu', \beta^\nu'; \gamma_0^\nu)_{\nu \in N} \rangle$ for $E'$, which also contradicts RE.

3. Suppose $\alpha_0^\nu < \frac{\alpha_0 + \beta_0}{N} < \alpha_0$. Note that by A1, $\alpha_0 > 0$. If $\frac{\alpha_0 + \beta_0}{N} \neq \alpha_0^\nu = \frac{\alpha_0 + \beta_0}{N}$, then a similar argument as in steps 1 and 2 above yields a contradiction. For instance, suppose that $\frac{\alpha_0 + \beta_0}{N} = \alpha_0^\nu < \frac{\alpha_0 + \beta_0}{N}$. If $\alpha_0^\nu < 1$, it is not difficult to construct another economy $E' = \langle N; (P, b); (\nu^\nu)_{\nu \in N} \rangle$ and its RS $\langle (p, w), (\alpha^\nu', \beta^\nu', \gamma_0^\nu)_{\nu \in N} \rangle$ such that $\alpha' + \beta' = \alpha + \beta$, and there exists one agent $\nu' \in N$ such that $\alpha_0^\nu' + \gamma_0^\nu' > \alpha_0^\nu$ and for any other $\nu \in N \setminus \{\nu'\}$, $\alpha_0^\nu' + \gamma_0^\nu' = \alpha_0^\nu$. Then, the desired contradiction follows from IND and RE. If $\alpha_0^\nu \geq 1$, then for a sufficiently large number $k > 1$, it is not difficult to construct an alternative, $k$-replica economy $\langle kN; (P, b); (\nu^\nu)_{\nu \in kN} \rangle$ and its RS $\langle (p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in kN} \rangle$ such that $\alpha' + \beta' = k(\alpha + \beta)$, and there exists one agent $\nu' \in kN$ such that $\alpha_0^\nu' + \gamma_0^\nu' < \alpha_0^\nu$ and for any other $\nu \in kN \setminus \{\nu'\}$, $\alpha_0^\nu' + \gamma_0^\nu' \leq \alpha_0^\nu$. Then, by IND and RI, this implies the violation of RE. Let $\frac{\alpha_0 + \beta_0}{N} = \alpha_0 = \frac{\alpha_0 + \beta_0}{N}$. If $N \geq 3$, then again applying IND and RE leads us to a desired contradiction if $N = 2$, it is possible to construct an alternative, $k$-replica economy $\langle kN; (P, b); (\nu^\nu)_{\nu \in kN} \rangle$ for sufficiently large number $k > 1$ and its RS $\langle (p, w), (\alpha^\nu; \beta^\nu; \gamma_0^\nu)_{\nu \in kN} \rangle$ such that $\alpha' + \beta' = \alpha_0$. If $N = 2$, then one can apply the above argument again with a suitable $k$.
that $\alpha^\nu_0 + \gamma^\nu_0 < \alpha^\nu_0$ and for any other $\nu \in kN \setminus \{\nu\}$, $\alpha^\nu_0 + \gamma^\nu_0 \leq \alpha^\nu_0$. Then, by IND and RI, this implies the violation of RE.

4. Suppose $\alpha^\nu_0 = \frac{\alpha_0 + \beta_0}{N} < \alpha^\nu_0$. It is possible to construct an alternative economy $\langle N; (P, b); (\omega^\nu_0)_{\nu \in N}\rangle$ and its RS $((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N})$ such that $\alpha' + \beta' = \alpha + \beta$, and there exists one agent $\nu' \in N$ such that $\alpha^\nu_0 + \gamma^\nu_0 < \alpha^\nu_0$ and for any other $\nu \in N \setminus \{\nu'\}$, $\alpha^\nu_0 + \gamma^\nu_0 \leq \alpha^\nu_0$. Then, the desired contradiction follows from IND and RE. Next, suppose $\alpha^\nu_0 < \frac{\alpha_0 + \beta_0}{N} = \alpha^\nu_0$. By A4, $\alpha^\nu_0 = \frac{\alpha_0 + \beta_0}{N} < 1$. Then, applying the same logic as in the case of $\frac{\alpha_0 + \beta_0}{N} - \alpha^\nu_0 \neq \frac{\alpha_0 + \beta_0}{N} - \alpha^\nu_0$, a contradiction follows from IND and RE.

Case 2: $\pi^{\text{max}} > 0$ and $\alpha + \beta \leq \sum_{\nu \in N} \omega^\nu = \omega$. If $p (\alpha + \beta) = p \omega$, then the same argument as for Case 1 can be applied, because $(\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N}$ is efficient. If $p (\alpha + \beta) < p \omega$, then by Lemma 2, there is a subset $N' \subset N$ such that for each $\nu \in N'$, $p \beta^\nu - w \beta^\nu = pb < \pi^{\text{max}} p \omega$ holds. Thus, by selecting $\omega^\nu \leq \omega^\nu$ appropriately from each $\nu \in N'$, it is possible to find $\omega' \equiv \sum_{\nu \in N \setminus N'} \omega^\nu + \sum_{\nu \in N'} \omega^\nu$ such that $p (\alpha + \beta) = p \omega$ holds. By Lemma 5, $((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N})$ constitutes an efficient RS for $E' = \langle N; (P, b); (\omega^\nu)_{\nu \in N}\rangle$. Then we can apply steps 1–4 of Case 1 above to check that any formulation of exploitation satisfying LES, RI, IND, and RE is identical to Definition 7 at $((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N})$ in $E'$. Then, by IND, the same statement also holds for $((p, w), (\alpha^\nu; \beta^\nu; \gamma^\nu)_{\nu \in N})$ in $E$.

Case 3: $\pi^{\text{max}} = 0$. Since $\pi^{\text{max}} = 0$, Proposition 3 implies that $\alpha^\nu_0 + \gamma^\nu_0 = \frac{\omega^\nu}{\omega}$ holds for each $\nu \in N$. This implies that $\alpha^\nu_0 + \gamma^\nu_0 = \frac{\alpha_0 + \beta_0}{N}$ holds for each $\nu \in N$ and this RS is efficient, according to Definition 10. By IND, suppose that $\omega = \sum_{\nu \in N} \omega^\nu + \sum_{\nu \in N} \beta^\nu$, $\omega^\nu = \frac{\omega}{N}$ for each $\nu \in N$, and $\sum_{\nu \in N} \alpha^\nu + \sum_{\nu \in N} \beta^\nu \in SP (N b)$, without loss of generality.

Let $c, \tau \in B (p, b)$ be the corresponding reference bundles. Because $p^{\text{max}} = 0$, then $p c - w c = \sum_{\nu \in N} \omega^\nu \leq pb - w \frac{\alpha_0 + \beta_0}{N} = 0$ and $p c - w c = \sum_{\nu \in N} \omega^\nu \leq pb - w \frac{\alpha_0 + \beta_0}{N} = 0$. Thus, $\alpha^\nu_0, \alpha^\nu_0 \geq \frac{\alpha_0 + \beta_0}{N}$ by $c, \tau \in B (p, b)$. By LES, $\alpha^\nu_0 \geq \alpha_0^\nu$ holds, so that either (i) every agent is neither an exploiter nor exploited, or (ii) every agent is an exploiter. Case (i) corresponds to the case that $\alpha^\nu_0 \geq \alpha^\nu_0 > \frac{\alpha_0 + \beta_0}{N} = \alpha^\nu_0 + \gamma^\nu_0$, for each $\nu \in N$, and it contradicts RE.

Let $\alpha^\nu_0 > \alpha^\nu_0 = \frac{\alpha_0 + \beta_0}{N}$. By Lemma IV, there exists an infinite sequence of economies $\{E^k\}_{k = 1}^{\infty}$ such that $E^k = \langle N; (P, b); (\omega^k)_{\nu \in N}\rangle$ for each $k = 1, \ldots, \infty$ and $(\omega^k)_{\nu \in N} \to (\omega^k)_{\nu \in N}$ as $k \to \infty$, and that there exists an infinite sequence of vectors $(\langle p^k, w^k \rangle, (\alpha^k; \beta^k; \gamma^k)_{\nu \in N})$, each of which con-
stitutes an efficient RS associate with \( \pi^{\text{max}}_k \) for \( E^k \) for each \( k = 1, \ldots, \infty \), such that \( \left( \pi^{\text{max}}_k, w^k, (\alpha^{kv}; \beta^{kv}; \gamma^{kv}_0)_{\nu \in \mathbb{N}} \right) \rightarrow \left( \pi^{\text{max}}, w, (\alpha^{kv}; \beta^{kv}; \gamma^{kv}_0)_{\nu \in \mathbb{N}} \right) \).

Then, the efficient RS \( \left( (p^k, w^k), (\alpha^{kv}; \beta^{kv}; \gamma^{kv}_0)_{\nu \in \mathbb{N}} \right) \) for \( E^k \), with associated reference bundles \( (\mathcal{E}^k, \mathcal{L}^k) \) has \( \pi^{\text{max}}_k > 0 \) for sufficiently large \( k \). By **Cases 1-2** above, it follows that \( \alpha^{\pi}_0 = \alpha^{\pi}_k \) for all \( k = 1, \ldots, \infty \). This implies a violation of **CON**, since \( \left( \pi^{\text{max}}_k, w^k, (\alpha^{kv}; \beta^{kv}; \gamma^{kv}_0)_{\nu \in \mathbb{N}} \right) \rightarrow \left( \pi^{\text{max}}, w, (\alpha^{kv}; \beta^{kv}; \gamma^{kv}_0)_{\nu \in \mathbb{N}} \right) \), but \( \left( \alpha^{\pi}_0, \alpha^{\pi}_k \right) \rightarrow (\alpha^{\pi}_0, \alpha^{\pi}_0) \).

In sum, if a definition of labour exploitation satisfies **LES**, **CON**, **RE**, **RI**, and **IND**, then the condition \( \alpha^\pi = \alpha^\pi_0 = \frac{\alpha^\pi + \beta^\pi}{N} \) must hold. □

### 7.3 Appendix 3: Additional Claims

In this Appendix, some additional claims made in the paper are rigorously proved. First, Example A.1 proves that, if the definition of exploitation satisfies **LES**, the **CECP** may hold if a definition of labour value is adopted which does not focus on profit-maximising processes.

**Example A.1:** Consider the following von Neumann technology:

\[
A = \begin{bmatrix} 2 & 4 & 2 & 2 \\ 4 & 4 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 8 & 6 & 6 \\ 12 & 12 & 6 & 0 \end{bmatrix}, \quad L = (1, 1, 1, 1),
\]

where the notation is the same as in Example 1 above and the production possibility set is \( P_{(A, B, L)} = \{ \alpha \in \mathbb{R}_- \times \mathbb{R}_+^2 : \exists x \in \mathbb{R}_+^4 : \alpha \leq (-Lx, -Ax, Bx) \} \).

Then, consider a convex cone subsistence economy defined by \( P_{(A, B, L)} \), \( b = (2, 2) \), and \( \omega = (N, N) \). Let \( (\omega^\nu)_{\nu \in \mathbb{N}} \) be such that \( \omega^\nu = (\delta^\nu, \delta^\nu) \), where \( \delta^\nu \leq 2 \), all \( \nu \in \mathbb{N} \), and \( \omega^i = (2, 2) \), some \( i \in \mathbb{N} \).

Let \( e_j \), \( j = 1, 2, 3, 4 \), be defined as in Example 1 discussed in Section 3 above. Then \( \hat{\alpha}^1 \equiv (0, 8) \), \( \hat{\alpha}^2 \equiv (4, 8) \), \( \hat{\alpha}^3 \equiv (4, 4) \), and \( \hat{\alpha}^4 \equiv (4, 0) \). Also, \( \hat{P}(\alpha_0 = 1) = co \{ \hat{\alpha}^1, \hat{\alpha}^2, \hat{\alpha}^3, \hat{\alpha}^4, 0 \} \). In this economy, \( p = (1, 0) \), \( w = 2 \) constitute a RS, with \( \alpha^\nu = 0, \beta^\nu = \frac{\delta^\nu}{2} \alpha^3 \), and \( \gamma^\nu_0 = 1 - \frac{\delta^\nu}{2} \), all \( \nu \in \mathbb{N} \). The corresponding aggregate production is \( \alpha + \beta = \frac{N\alpha^3}{2} \). At this RS, \( \pi(\alpha + \beta; p, w) \equiv \frac{p\alpha^3 - w\alpha^3}{p\alpha^3} \equiv 1 \), whereas \( \pi(\alpha^1; p, w) \equiv \frac{p\alpha^3 - w\alpha^3}{p\alpha^3} < 0 \), \( \pi(\alpha^2; p, w) \equiv \frac{p\alpha^3 - w\alpha^3}{p\alpha^3} = 1 \), \( \pi(\alpha^3; p, w) \equiv \frac{p\alpha^3 - w\alpha^3}{p\alpha^3} = 1 \). Thus, \( \hat{P}(p, w) = \{ \alpha \in P \mid \exists \lambda > 0 : \lambda \alpha \in co \{ \alpha^3, \alpha^4 \} \} \).
Choose \( \tau = \zeta = \frac{\alpha^2}{2} \), so that \( \tau, \zeta \in B(p, b) \). Since \( \frac{\pi^2 - w_0^2}{2} = 1 = \pi_{\max} p_{\lambda_{\min}} = \pi_{\max} p_{\lambda_{\max}} = \pi_{\max} p_{\alpha^2} \), Theorem 3 states that the CECP holds if a definition of exploitation is adopted which satisfies LES with \( \tau = \zeta = \frac{\alpha^2}{2} \). However, since \( \pi \left( \frac{\alpha^2}{2}; p, w \right) = \frac{1}{2} \), it follows that \( \alpha^2 = \alpha^2 \notin \mathcal{P} (p, w) \). □

Example A.2 instead proves the existence of inefficient RS’s, and it highlights their structure and the role of big capitalists in generating inefficiencies.

**Example A.2:** Consider the following von Neumann technology:

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}, \quad L = (1, 0.8),
\]

where the notation is the same as in Example 1 above and the production possibility set is \( P_{A,B,L} \equiv \{ \alpha \in \mathbb{R}_+ \times \mathbb{R}_+ ^2; \exists x \in \mathbb{R}_+ ^2 : \alpha \geq (-Lx, -Ax, Bx) \} \).

Consider a subsistence economy \( E = \langle N; (P, b); (\omega^v)_{v \in N} \rangle \) defined by \( N = \{1, 2\} \), \( P = P_{A,B,L} \), \( b = (1, 1) \), and \( (\omega^1, \omega^2) = ((2, 1), (0, 0)) \).

To begin with, it is shown that the price vector \( (p, w) = ((1, 0), 1) \), and the allocation \( (\alpha^1; \beta^1; \gamma_0^1) = (0; (-1, -1, -1, (3, 3)); 0) \), and \( (\alpha^2; \beta^2; \gamma_0^2) = (0; 0; 1) \) constitute a RS for this economy. First, given this \( (p, w) \), the activity \( \beta^1 = (-1, -1, (3, 3)) \in P_{A,B,L} \) is a maximal profit-rate production point. Note that \( \pi^1 (p, w) \equiv \frac{p\beta^1 - w\beta_0^1}{p\beta^1} = 1. \) Let \( \beta' \equiv (-0.8, -2, 1, 4, 4) \in P_{A,B,L} \). Then, \( \pi' (p, w) \equiv \frac{p\beta' - w\beta_0^1}{p\beta^1} = 0.6 \). Thus, \( \pi^1 (p, w) > \pi' (p, w) \). By the property of \( P_{A,B,L} \), any other production point \( \alpha \in P_{A,B,L} \) is represented as \( \alpha \leq t \beta^1 + t\beta' \) for some suitable non-negative values \( t, t' \geq 0 \). Thus, \( \pi^1 (p, w) > \pi' (p, w) \) implies that \( \beta^1 \) is a maximal profit-rate point. Second, since \( pb = 1 \) and \( w = 1 \), agent 2’s optimal solution is \( \gamma_0^2 = 1 \), since \( \omega^2 = (0, 0) \). Third, for agent 1, \( (0; \beta^1; 0) \) is the optimal solution, since \( p\beta_0^1 - w\beta_0^1 = pb \), \( \pi^1 (p, w) = \pi_{\max} \), and \( p\beta_0^1 < p\omega^1 \). Note that \( \beta^1 = 2b \) and \( \beta_0^1 = \gamma_0^2 = 1 \). Since \( \beta^1 \leq \omega^1 \), which implies \( \beta^1 \leq \omega \equiv \omega^1 + \omega^2 \), \( (p, w), (\alpha^1; \beta^1; \gamma_0^1) \) is a RS with \( p\beta_0^1 < p\omega \).

The allocation \( (\alpha^1; \beta^1; \gamma_0^1) \in N \) is inefficient. Consider \( \beta^1 \) as the alternative social production point. Note that \( \beta^1 = \omega, \beta^1 = (2, 3) \geq 2b \), and \( \beta_0^1 = 0.8 < \beta^1 \). Construct an alternative allocation \( (\alpha^2; \beta^2; \gamma_0^2) \in N \) as \( (\alpha^1; \beta^1; \gamma_0^1) = (0; \beta^1; 0) \) and \( (\alpha^2; \beta^2; \gamma_0^2) = (0; 0.8) \). Since \( \gamma_0^2 < \gamma_0^3 \), this implies that the RS allocation \( (\alpha^2; \beta^2; \gamma_0^2) \) is not efficient. □
As noted in Section 5 above, the source of the inefficiency is the violation of the assumption of local nonsatiation: an increase in wealth does not make agent 1 better off. Thus, it is individually rational for agent 1 to use the labour intensive activity $\beta^1$ because it yields the maximum rate of profit at $(p, w) = ((1, 0), 1)$, instead of the capital-intensive, and socially optimal technique $\beta^0$ which yields a lower profit rate.

8 References


