Quotas under dynamic Bertrand competition

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Abstract

We present a dynamic model of Bertrand competition, where a quota is treated as an intertemporal constraint, rather than as a capacity constraint, limiting total quantity of exports during a given period but allowing an exporter to strategically vary the rates of exports over time. We show that a quota results in higher prices than a tariff of equal imports. We also find that firms never play mixed strategies in equilibrium, which contrasts sharply from the result from a one-shot game, in which the only equilibrium under a quota is in mixed strategies (Krishna 1989).

***JEL* Code**: F13

Keywords: Tariffs vs. quotas, dynamic Bertrand competition

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1. Introduction

This paper presents a new model of dynamic Bertrand competition under a quota. The model reflects the idea that an exporter facing a quota behaves quite differently from a firm facing a capacity constraint. A quota specifies a limit to how much a firm is allowed to export in a given period, say, one year. Thus, a firm under a quota can still vary the rates of exports during the period as long as total sales are below the limit. In contrast, a capacity-constrained firm is subject to the production limit per unit of time, say, due to the plant size, and therefore does not face a dynamic constraint like a firm under a quota. This distinction between a quota and a capacity constraint is lost in the literature, however. Practically every dynamic model of oligopoly examining the effect of quotas, no doubt influenced by the capacity constraint literature (e.g., Kreps and Sheinkman, 1983), treats a quota as a capacity constraint.

The objective of the present paper is to explore the nature of a quota as a dynamic quantity constraint. To that end, we consider duopolistic competition between a foreign firm and a domestic firm in the domestic market during a given period. Firms produce differentiated goods, and choose prices continuously during this period to maximize total profits. In addition, the foreign firm is subject to a quota during that period. We solve this model for a subgame-perfect equilibrium.

Our analysis makes two main contributions to the literature. A first is to the literature on the relative effect of tariffs and quotas. It is well know that tariffs and quotas are equivalent under perfect competition in the sense that replacement of a tariff by an equal-import quota leaves the domestic price intact. Over the last quarter-century the validity of this proposition has been examined under various forms of market structure.[1] In particular, Itoh and Ono (1982, 1984), Harris (1985), and Krishna (1989) showed for Bertrand duopoly that the quota results in a higher domestic price than the tariff. On the other hand, Hwang and Mai (1988) reaffirmed the equivalence for Cuournot duopoly.

The relative effect of tariffs and quotas has also been examined in dynamic contexts. Rotemberg and Saloner (1989) analyze a repeated-game setting and find that a quota results in a lower domestic price than a tariff of equal imports because a quota makes collusion more difficult to sustain. Considering a dynamic Cournot model with slow price adjustment, Dockner and Haug (1990) show that a quota results in higher prices than the quota. Thus, they show that the static result of Hwang and Mai (1988) does not extend to a dynamic setting, but do not explain why the market is out of equilibrium in the first place. Miyagiwa and Ohno (1995, 1999) re-examine the infant-industry argument for protection, showing that a quota generally delays adoption or innovation of new technology relative to a tariff. All these authors assume the quota binding at each point in time, thereby ignoring the possibility of intra-period variations in prices, which is the focus of this chapter. The exception is the work of Miyagiwa and Ohno (2001), which treats quotas as dynamic constraints and examines the intra-period variations in prices and output for Cournot duopoly.

The second contribution this paper makes to the literature concerns the fact that models of oligopoly under a quota may not possess equilibria in pure strategies. A basic reference is Krishna (1989), who shows that for differentiated Bertrand competition the

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[1] See, e.g., Helpman and Krugman (1989) and Brander (1995) for surveys of early work.

only Nash equilibrium under the quota is in mixed strategies, with the domestic firm randomizing between two prices and the foreign firm playing a pure strategy. In repeated-game setting, Rotemberg and Saloner (1989) show that the stage game has homogeneous-goods Bertrand firms playing mixed strategies in equilibrium. However, the absence of pure-strategy equilibrium under quotas is not unique to Bertrand competition. Reitzes and Grawe (1994) show that Cournot duopoly has the equilibrium only in mixed strategies under market-share quotas, i.e., when the foreign firm is constrained not to exceed the prescribed fraction of sales in the domestic market.

The logic underling these results cannot be faulted. Some researchers have nonetheless expressed skepticism about mixed-strategy equilibriums. The following statement by Brander (1995, p. 1436) summarizes this sentiment: "The value of Krishna is not so much that it is likely to be a literal description of an actual outcome. The paper's contribution is that it focuses attention on the idea that a VER (and, by extension, any trade policy instrument) can have important effects through the effects on imperfectly competitive rivalries between firms." Our model throws light on this issue. The above-mentioned models possess no pure-strategy Nash equilibrium under the quota because the game is static (Krishna 1989, Reitzes and Grawe 1994) or equivalently a quota is binding at every instant (Rotemberg and Saloner 1989). Using Krishna (1989)'s model as a benchmark, we show that when a quota serves as a dynamic constraint, firms never play mixed strategies.

The next section reviews the Krishna model, which serves as a benchmark. Section 3 sets out the dynamic Bertrand model, showing the absence of mixed strategy equilibrium under a quota. Section 4 presents additional properties of the equilibrium, and shows that normally a quota leads to higher price trajectories than the equal-import tariff. Section 5 reconsiders the facilitating nature of quotas. Concluding remarks are in the final section.

2. The one-shot price-setting game under the quota

This section provides a quick review of the Krishna (1989) model, pointing out its essential features that will be useful in the next section. The Krishna model is just an application of the standard differentiated-goods Bertrand duopoly model. Thus, suppose that a domestic and a foreign firm play a one-shot price-setting game in the domestic market. Let P and p denote the prices of the domestic and the foreign good, respectively. Write the demand function for the domestic and the foreign good as X(P, p) and x(P, p), respectively. Assume that demands are downward sloping ($X_P < 0$ and $x_p < 0$), and goods are substitutes ($X_p > 0$ and $x_P > 0$), where the subscripts denote partial derivatives.

Suppose that free trade prevails. The foreign firm takes P as given and chooses p to maximize the profit:

$$v(P, p) \equiv px(P, p) - c[x(P, p)],$$

where c(.) is the foreign firm's cost function, with c' > 0 and c" ≥ 0 (primes denote derivatives). Similarly, the domestic firm takes p as given and maximizes the profit:

$$V(P, p) \equiv PX(P, p) - C[X(P, p)],$$

where C(.) is the domestic firm's cost function, with C' > 0 and C" ≥ 0. The first-order condition:

(1) $$v_p(P, p) = x(P, p) + px_p(P, p) - c'[x(P, p)]x_p(P, p) = 0$$

implicitly defines the foreign firm's (static) best-response function, $p = b^S(P)$. The domestic firm's (static) best-response function, $P = B^S(p)$, obtains analogously from the first-order condition:

(2) $$V_P(P, p) = X(P, p) + PX_P(P, p) - C'[X(P, p)]X_P(P, p) = 0.$$

Assume that $v(P, p)$ is strictly concave in p and $V(P, p)$ is strictly concave in P so that the best-responses are indeed "functions".

Solving the two best-response functions simultaneously yields the static or one-shot Nash equilibrium, denoted by (P^S, p^S). (The superscript s indicates the "static" equilibrium). The best-response functions and the Nash equilibrium (P^S, p^S) under free trade are shown in Figure 1. The best-response functions slope upward since prices are strategic complements as we assume. The usual conditions are invoked to ensure the uniqueness of the equilibrium.[2] The equilibrium outputs and profits readily obtain by substituting P^S and p^S into each firm's demand and objective functions. Let $X^S \equiv X(P^S, p^S)$ and $x^S \equiv x(P^S, p^S)$ denote, respectively, the domestic and the foreign firm's static equilibrium output.

Suppose the quota is set at $\tilde{q}$. The effect is seen in Figure 1.[3] The curve qq* represents the locus of P and p such that the demand for the foreign product just satisfies the quota; that is, $\tilde{q} = x(P, p)$. Given P, if the foreign firm charges a price less than the

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[2] $|v_{pp}| > |v_{pP}|$ and $|V_{PP}| > |V_{Pp}|$ guarantee the equilibrium to be unique and stable.

price that satisfies this equation, the demand for the foreign product exceeds the quota. But the foreign firm is not allowed to set a price below the price implied by the locus qq*. Therefore, its best-response function consists of the part of qq* to the right of the b^s schedule and the part of b^s below the locus qq*, as indicated by the thick line with a distinct kink at S. The domestic firm's best-response function under the quota (also shown in bold) is not even continuous, consisting of the part of its original best-response curve B^s to the right of $\hat{p}$ and the horizontal line at P^H up to $\hat{p}$.

Since the two best-response functions do not intersect each other, there is no Nash equilibrium in pure strategies. However, there is a unique Nash equilibrium in which the domestic firm randomizes between P^H and P^L and the foreign firm charges $\hat{p}$. Since both P^H and P^L are higher than P^s, and $\hat{p}$ is higher than p^s, we conclude that the quota raises prices of both firms. This conclusion holds even when there is a tariff initially, and when the quota is set equal to the volume of imports under the tariff. That is, the quota raises prices of both the foreign and the domestic firm relative to the tariff.

Notice that, when the domestic firm chooses P^H, the demand for imports exceeds the quota, thereby creating a shortage of imported goods. This shortage is assumed to vanish through costless arbitrage among consumers. That is, consumers lucky enough to get the imports at $\hat{p}$ make profits by selling them at p^A, the market-clearing price. The effect on the domestic firm's demand therefore is exactly what it would be if the foreign

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[3] Figure 1 duplicates Krishna's Figure 3, where the quota is set at the free-trade level of imports.

firm charged p^A instead of $\hat{p}$; i.e., the demand for the domestic good is given by $X(P^H, p^A)$.

Notice also that when the domestic firm charges P^L the quota is not binding. Finally, since the domestic firm randomizes between P^H and P^L, its profit from each action must be identical, as indicated by the iso-profit curve VV*.

3. The quota as a dynamic constraint

3.1 Setup

We now consider the model of differentiated-goods Bertrand duopoly in a infinite-time setting, where time t flows continuously from zero to infinity. The time set $[0, \infty)$ is divided into infinitely many non-overlapping intervals, called years, of the form $[\tau, \tau + 1)$, where $\tau = 0, 1, 2, ...$ $X[P(t), p(t)]$ and $x[P(t), p(t)]$ now denote instantaneous demand functions at time t. The demand functions are stationary over time but values of the functions vary as prices change over time. Finite reservation prices exist and quantities demanded are finite for all (p, P). The Markov-perfect equilibrium is used as the solution concept so the properties of the one-shot game are preserved.

Under free trade firms choose prices simultaneously at each time to maximize profits over an infinite time horizon. Since there are no “state variables”, the model is stationary in the sense that subgames starting at any $t \in [0, \infty)$ are identical. Therefore, the equilibrium outcome at any t is exactly the same as that in the static price-setting game. That is, in the Markov-perfect equilibrium the foreign firm produces x^s units and charges p^S while the domestic firm produces X^S units and sells them at p^S at each $t \in [0, \infty)$.

3.2 Quotas

Suppose that the foreign firm is subject to the quota such that its yearly output (and sale) is not allowed to exceed the limit $\bar{q}$ per year. Since a year is a unit interval, we can write the constraint facing the foreign firm as:

$$\int_{\tau}^{\tau+1} x[P(t), p(t)]dt \leq \bar{q}, \tag{3}$$

for all $\tau = 0, 1, 2, ...$ To make the analysis interesting, we assume that the quota is binding:

$$\bar{q} \leq \int_{\tau}^{\tau+1} x^S dt.$$[4]

The constraint (3) implies that how much more output the foreign firm can sell during the remainder of any given year depends on the total output it has sold to date within that year. Thus, the past sales become a "state variable" in the dynamic optimization problem for the remainder of the year. Notice however that this state variable is not carried over to the future years since the foreign firm cannot carry the unused portion of the quota over to the following year. Then, by the property of Markov-perfect equilibrium, whatever

[4] The quota slightly exceeding the free-trade level still can affect the equilibrium.

happens during any given year has no effect on the subgames beginning in the following or later years. This fact lets us focus on the first year; i.e., $t \in [0, 1)$.[5]

Since we are interested in the subgame perfect equilibrium we solve the model backward, using dynamic programming techniques. Suppose we are at a given time $z \in [0, 1)$. Let $\mu(z)$ denote the foreign firm's total output up to a time $z < 1$, that is:

$$\mu(z) \equiv \int_0^z x[[P(t), p(t)]]dt.$$

Then, the maximum quantity the foreign firm is allowed to sell in the interval $[z, 1)$ is equal to $\bar{q} - \mu(z)$. Thus, the foreign firm's quantitative constraint at time z is written as:

$$\int_z^1 x[P(t), p(t)]dt \leq \bar{q} - \mu(z). \tag{4}$$

Subject to this constraint, the foreign firm takes $\mu(z)$ and $P(z)$ as given and chooses $p(t)$ for $t \in [z, 1)$ to maximize:

$$\int_z^1 e^{-rt}\Big\{p(t)x[P(t), p(t)] - c\{x[P(t), p(t)]\}\Big\}dt$$

where r is the instantaneous rate of interest. The domestic firm faces no quantitative restriction so it takes $p(z)$ as given and chooses $P(t)$ to maximize the following intertemporal profit over $t \in [z, 1)$:

[5] Since there are no state variables linking years, the equilibrium outcome in year one also holds in any other year in which the quota is in effect.

$$\int_{z}^{1} e^{-rt}\Big\{P(t)X[P(t), p(t)] - C\{X[P(t), p(t)]\}\Big\}dt.$$

3.3 Characterization of the equilibrium under the quota

We first prove the following proposition.

Proposition 1. Both the firms play pure strategies at each $t \in [0,1)$.

Proof is by contradiction. Suppose that a mixed strategy is played at any time $t^0 \in [0, 1)$. When the quantitative restriction does not bind, strict concavity of the flow profit functions V(p, p) and v(P, p) implies that firms play pure strategies. Therefore, for firms to play mixed strategies at t^0, at least one firm must have a non-concave flow profit function. The discussion developed in section 2 leads to the following result.

Result: When the quota binds at t^0, at a time $t \in [t^0, t^0 + \varepsilon]$ where ε (> 0) is arbitrarily small, the domestic firm randomizes between $\bar{P} \equiv \bar{P}(t^0)$ and $\underline{P} \equiv \underline{P}(t^0)$ while the foreign firm charges $\hat{p} \equiv \hat{p}(t^0)$.[6]

If so, the foreign firm reaches the quantitative limit $\bar{q}$ by the time $t^0 + \varepsilon$, and will make no more sales after that. This will allow the domestic firm to charge the monopoly price P^m for the rest of the year. On the other hand, if the domestic firm plays $\bar{P}$ the quota will not be used up in $[t^0, t^0 + \varepsilon]$.

Let $\Omega(P, \hat{p}, t^0)$ denote the domestic firm's future profit when it charges the price P during $[t^0, t^0 + \varepsilon]$. Since it randomizes between $\bar{P}$ and $\underline{P}$, the domestic firm's future profits must be identical; that is, $\Omega(\bar{P}, \hat{p}, t^0) = \Omega(\underline{P}, \hat{p}, t^0)$. We show that this is a contradiction.

Begin with the case in which the domestic firm charges $\bar{P}$ during $[t^0, t^0 + \varepsilon]$. Since the restriction is binding and consumer arbitrage takes place, the effect on the domestic firm's demand is exactly what it would be if the foreign firm charged some arbitrage price p^A $(\geq \hat{p})$.[7] Thus, during the interval $[t^0, t^0 + \varepsilon]$ the domestic firm makes the profit:

$$\int_{t^0}^{t^0+\varepsilon} e^{-r(t-t^0)}\{\bar{P}X(\bar{P}, p^A) - C[X(\bar{P}, p^A)]\}dt$$

$$= \varepsilon\{\bar{P}X(\bar{P}, p^A) - C[X(\bar{P}, p^A)]\},$$

where we used the fact that for small ε

$$\int_{t}^{t+\varepsilon} f(z)dz = f(t)\varepsilon.$$

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[6] Strict concavity of the profit function rules out other types of mixed-strategy equilibria.

[7] The actual value of p^A is different from that in Section 2.

After $t^0 + \varepsilon$, there are no more sales by the foreign firm since the quota has been used up. The domestic firm therefore earns the monopoly profit for the rest of the year. Let $\Phi^m(t^0 + \varepsilon)$ denote the discounted sum of the monopoly profits earned in the interval $[t^0 + \varepsilon, 1)$. Now, we can write the future profit from charging $\bar{P}$ as:

$$\Omega(\bar{P}, \hat{p}, t^0) = \varepsilon\{\bar{P}X(\bar{P}, p^A) - C[X(\bar{P}, p^A)\} + \Phi^m(t^0 + \varepsilon).$$

Consider next the case in which the domestic firm charges $\underline{P}$ in $[t^0, t^0 + \varepsilon]$. The domestic firm's profit during this short interval is given by

$$\varepsilon\{\underline{P}X(\underline{P}, \hat{p}) - C[X(\underline{P}, \hat{p})]\}.$$

In this case the foreign firm will not use up its quota by time $t^0 + \varepsilon$, and continue to produce after the time $(t^0 + \varepsilon)$. Let $\Phi(t^0 + \varepsilon)$ denote the discounted sum of profits the domestic firm will earn in the interval $[t^0 + \varepsilon, 1)$. Then, the domestic firm's future profit is given by:

$$\Omega(\underline{P}, \hat{p}, t^0) = \varepsilon\{\underline{P}X(\underline{P}, \hat{p}) - C[X(\underline{P}, \hat{p})]\} + \Phi(t^0 + \varepsilon).$$

Notice that

$$\Phi(t^0 + \varepsilon) < \Phi^m(t^0 + \varepsilon)$$

since $\Phi^m(t^0 + \varepsilon)$ sums up all the monopoly profits between $t^0 + \varepsilon$ and 1.[8] Therefore, for ε arbitrarily small, we have

$$\Omega(\bar{P}, \hat{p}, t^0) > \Omega(\underline{P}, \hat{p}, t^0),$$

implying that the domestic firm strictly prefers charging $\bar{P}$ to charging $\underline{P}$. But this contradicts the initial hypothesis that the domestic firm randomizes between $\bar{P}$ and $\underline{P}$ in the interval $[t^0, t^0 + \varepsilon]$. Since t^0 is arbitrary, this contradiction proves that the domestic firm does not randomize at $t \in [0, 1)$.[9] To complete the proof, we must show that the quota is not used up at $t < 1$ in a pure-strategy equilibrium, either. To see this, suppose the contrary; that is, suppose that the foreign firm makes the final sale of x units at $t < 1$. Should it postpone the sale of one unit of output to $t + \varepsilon$ instead, the foreign firm would lose $p'(x)x + p(x) - c'(x)$ in profit at time t but would gain $p(0) - c'(0)$ at time $t + \varepsilon$. For an arbitrarily small ε, therefore, such postponement would always increase the foreign firm's intertemporal profit. The proof of Proposition 1 is now complete.

As we already noted, a mixed-strategy equilibrium outcome emerges only at the moment when the quota is all used up. But Proposition 1 says that a mixed-strategy outcome is never observed at $t < 1$. These two results lead to the next proposition.

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[8] The difference between $\Phi^m(t^0 + \varepsilon)$ and $\Phi(t^0 + \varepsilon)$ may be small if the quota is used up shortly after $t^0 + \varepsilon$ or if the rates of exports are low. However small, the difference, $\Phi^m(t^0 + \varepsilon) - \Phi(t^0 + \varepsilon)$, does not vanish when ε goes to zero, which is sufficient to produce a desired contradiction.

[9] This result is not affected by the size of r ($< \infty$).

Proposition 2: In the equilibrium the foreign firm does not use up its quota before the end of the year. That is:

$$\mu(t) < \bar{q} \text{ for all } t < 1$$

and

$$\lim_{t \to 1} \mu(t) = \bar{q}.$$

4. Equilibrium paths under the quota

Now we solve the model. Propositions 1 and 2 let us focus on a pure-strategy equilibrium. We use dynamic programming techniques to obtain a subgame-perfect equilibrium.[10]

Let $\tilde{P}[t; j, \mu(j)]$ and $\tilde{p}[t; j, \mu(j)]$ denote the equilibrium paths for the subgame starting at time j (< 1) with the initial condition that the foreign firm has sold $\mu(j)$ ($< \bar{q}$) between [0, j). Then the intertemporal equilibrium profits for this subgame are given by:

$$\Pi[j, \mu(j)] \equiv \int_j^1 e^{-rz}\{\tilde{P}X(\tilde{P}, \tilde{p}) - C[X(\tilde{P}, \tilde{p})]\}dz,$$

for the domestic firm and,

$$\pi[j, \mu(j)] \equiv \int_j^1 e^{-rz}\{\tilde{p}x(\tilde{P}, \tilde{p}) - c[x(\tilde{P}, \tilde{p})]\}dz,$$

for the foreign firm. Now divide the interval [t, 1) into two disjoint subintervals [t, t + Δt) and [t + Δt , 1), where Δt is small. Assume that the optimization problem is solved for the second subinterval, with the maximum profit from that subinterval denoted by Π[t + Δt, μ(t + Δt)]. Then, by the principle of optimality, we can express the domestic firm's optimization problem for the interval [t, 1) as:

(5) $$\max_P \left\{ \int_t^{t+\Delta t} e^{-rz}\{P(z)X[P(z),\ p(z)] - C\{X[P(z),\ p(z)]\}dz + \Pi[t + \Delta t, \mu(t + \Delta t)] \right\}.$$

Note that $\mu(t + \Delta t) = \mu(t) + \Delta\mu$. Since Δt is small, P(z) can be treated as a constant in (5). Also, for small Δt we have:

(6) $$\int_t^{t+\Delta t} f(z)dz = f(t)\Delta t.$$

Using (6) on the first term of (5) and expanding the second term by a Taylor-series expansion, we can rewrite (5) as:

(7) $$\max_P \left\{\Delta t\{PX(P, p) - C[X(P, p)]\}e^{-rt} + \Pi[t, \mu(t)] + \Pi_t[t, \mu(t)]\Delta t + \Pi_\mu[t, \mu(t)]\Delta\mu + \text{h.o.t.}\right\},$$

[10] See Kamien and Schwartz (1981, chapter 20) for an exposition of dynamic programming techniques.

where the h.o.t. represents all the higher-order terms in the Taylor-series expansion.

Since the second and the third term in (7) do not depend on P, they can be ignored in the optimization problem. Then, dividing through the remaining terms by Δt and taking the limit $\Delta t \to 0$, we can rewrite the domestic firm's objective function as

$$\max \{ e^{-rt}\{PX(P, p) - C[X(P, p)]\} + \Pi_\mu[t, \mu(t)]x(P, p)\}, \tag{8}$$

where x(P,p) is used to replace $\mu'(t)$.[11] The first-order condition to (8) is given by

$$\Psi \equiv e^{-rt}\{X(P, p) + PX_P(P, p) - C'[X(P, p)]X_P(P, p)\} + \Pi_\mu[t, \mu(t)]x_P(P, p) = 0, \tag{9}$$

which defines the domestic firm's best-response function B(p, t) at time t.

The first-order condition (9) has the following interpretation. The first term on the right measures the change in the domestic firm's profit at time t due to a small increase in P. Raising P also induces an instantaneous increase in imports ($x_P > 0$), which reduces the remaining quota through the increase in $\mu(t)$, which in turn raises the domestic firm's profits from sales made during the remainder of the year ($\Pi_\mu > 0$). This dynamic effect of a price change is captured in the second term on the right-hand side of (9). For an optimum the domestic firm chooses P to equate these two marginal effects on profits.

The properties of the best-response function B(p, t) can now be examined. We first compare B(p, t) with the best-response function $B^S(p)$ from the static game of Section 2. Notice that in the first term on the right-hand side of (9) the expression in

[11] By definition, $\mu'(t) = x(P, p)$. Note also that the h.o.t. vanishes when the limit is taken.

braces is the same as in the first-order condition of the static game [see eq. (2)]. On the other hand, by the preceding argument the second term on the right-hand side of (9) is positive. Thus, (2) and (9) imply that for a given p the domestic price must be higher in the dynamic game than in the static game; i.e., $P(t) = B(p, t) > B^S(p)$. Graphically, $B(p, t)$ lies "outside" $B^S(p)$.

We next examine how the domestic firm's best-response function shifts over time. Differentiating (9), while holding p constant, we obtain

$$dP/dt = - \frac{(\Pi_{\mu t} + \Pi_{\mu\mu}x)x_P + r\Pi_\mu x_P}{e^{-rt}(2X_P + PX_{PP} - C''X_P^2 - C'X_{PP}) + \Pi_\mu x_{PP}} \quad . \tag{10}$$

The denominator is negative by the second-order condition. Lemma 1 in the Appendix shows that $(\Pi_{\mu t} + \Pi_{\mu\mu}x)x_P$ is positive. Since Π_μ and x_P are also positive, we have $dP/dt > 0$, meaning that the domestic firm's best-response function shifts out over time. To sum the result so far, the domestic firm's dynamic best-response function under the quota is located outside of its static-game (and hence free-trade) best-response function, and shifts out over time.

Turning to the foreign firm, we write its optimization problem as

$$\max_p \left\{ \int_t^{t+\Delta t} e^{-rz}\{p(z)x[P(z), p(z)] - c\{x[P(z), p(z)]\}dz + \pi[t + \Delta t, \mu(t + \Delta t)] \right\} .^{12} \tag{11}$$

[12] By definition π is the equilibrium profit under the quota, so it subsumes the quota constraint (4).

The procedure used in the case of the domestic firm can be applied to simplify (11) to the following:

$$\max_{p} \left\{e^{-rt}\{px(P, p) - c[x(P, p)]\} + \pi_{\mu}[t, \mu(t)]x(P, p)\right\}.$$

The first-order condition is

$$(12) \qquad \psi \equiv e^{-rt}\{x(P, p) + px_p(P, p) - c'[x(P, p)]x_p(P, p)\}$$
$$+ \pi_{\mu}[t, \mu(t)]x_p(P, p) = 0,$$

which yields the foreign firm's best-response function, $p = b(P, t)$, at time $t < 1$.

How does $b(P, t)$ compare with the best-response function $b^S(P)$ from the one-shot game in Section 2? The expression in the first term on the right-hand side of (12) is the same as that in the first-order condition in the static game [see eq. (1)] so the position of $b(P, t)$ relative to $b^S(P)$ depends on the second term, which captures the following dynamic effect. A small increase in p decreases the demand for exports ($x_p < 0$), thereby relaxing the quota constraint for the remainder of the year and affecting the foreign firm's future profit by π_{μ}.

Thus, the term ($-\pi_{\mu}$) may be interpreted as the dynamic shadow price of the quota.[13] The presumption is that this shadow price is positive (i.e., $\pi_{\mu} < 0$) when the quota is binding. The reason is that an increase in μ means the foreign firm has less to export for the rest of year and hence earns a smaller profit. We treat this as the normal

[13] See Kamien and Schwartz (1981, p. 240) for this interpretation.

case. Then, the second term on the right-hand side of (12) is positive, implying that $p = b(P, t) > b^S(P)$; that is, for a given P, the domestic firm charges a higher price than it does in a static game.

Note, however, that a quota softens price competition and increasing profits. If this beneficial effect of a quota dominates the adverse effect described above, π_μ is positive or the shadow price negative. In such a case the second term on the right-hand side of (12) is negative, and hence $p = b(P, t) < b^S(P)$; i.e., the $b(P, t)$ schedule lies inside the $b^S(P)$ schedule. This surprising phenomenon may occur only if the quota is not very restrictive, and only early in the year. No matter what the size of a (binding) quota, π_μ turns negative towards the end of the year when the foreign firm used up most of its quota and the remaining quota gets tighter.[14]

We next examine the transition of the best-response function over time. Differentiating (12), while holding P constant, yields:

$$dp/dt = -\frac{(\pi_{\mu t} + \pi_{\mu\mu}x)x_p + r\pi_\mu x_p}{e^{-rt}(2x_p + px_{pp} - c''x_p^2 - c'x_{pp}) + \pi_\mu x_{pp}}\ .$$

The denominator is negative by the second-order condition. Lemma 2 in the appendix shows that the first term of the numerator is positive. Thus, the direction of shift again depends on the sign of $\tilde{\pi}_\mu$ In the normal case, π_μ is negative, so $dp/dt > 0$. Thus, the

[14] Proof of this claim is as follows. By the Mean-Value Theorem $\pi(t, \bar{q} - \gamma) = \pi(t, \bar{q}) - \pi_\mu(t, q^*)\gamma$, where $q^* \in [\bar{q} - \gamma, \bar{q}]$ and $\gamma > 0$. The term on the left is the foreign firm's profit from exporting $\bar{q} - \gamma$ units of output between time t and 1, and is positive by Proposition 1. The first term on the right is the profit when

foreign firm's best-response function shifts out over time. If π_μ is positive, however, we cannot rule out the possibility that the foreign firm's best-response function shifts in for some time.

Solving the best-response functions (9) and (12) simultaneously, we can find the equilibrium prices at each t. Let P*(t) and p*(t) denote the equilibrium outcome for the whole game; that is, $P^*(t) \equiv \tilde{P}\,[t; 0, \mu(0)]$ and $p^*(t) \equiv \tilde{p}\,[t; 0, \mu(0)]$ for $t \in [0, 1)$.

We now present some properties of the equilibrium price paths. We already established that under the quota the domestic firm's best-response function is located outside its static-game counterpart, $B^s(p)$, and shifts out over time. We then proved that in the normal case the foreign firm's best-response function also is placed outside the best-response function $b^s(P)$ and shifts out after some $t \geq 0$. We also showed that no matter what the size of a quota, we must have the normal-case result towards the end of the year. These three facts lead to the following general result.

Proposition 3: There is an interval $[\alpha, 1) \subseteq [0, 1)$ in which the equilibrium prices are higher than those under free trade and increasing over time:

$$P^*(t) > P^s,\ p^*(t) > p^s,\ \ P^{*\prime}(t) > 0, \text{ and } p^{*\prime}(t) > 0 \text{ for all } t \in [\tau, 1).$$

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no quota remains and hence identically equals zero. Thus, for the right-hand side to be positive, π_μ(t, q*) must be negative.

If the quota is sufficiently restrictive relative to free trade, the conclusion of proposition 3 holds for the entire period. If the quota is not very restrictive, however, π_μ may be positive so that the foreign firm's best-response function may be inside the position of its counterpart from the one-shot game, and shifting in. Thus, we cannot rule out the possibility that the equilibrium prices fall below those under free trade, and decreasing for some time of year. But if at all, this occurs early in the year.

We now ask what happens when the home government imposes a more restrictive quota. A tighter quota raises the shadow price of the quota at each point in time, thereby inducing the foreign firm to raise its price at $t \in \tilde{\square}$ ı. Further, since prices are strategic complements, the domestic price also rises throughout the period. We state this in the next proposition, while relegating the proof to the appendix (see lemma 3 there).

Proposition 4: Decreasing the quota raises both firms' price trajectories.

Propositions 3 and 4 lead to the following result.

Proposition 5: When the quota is sufficiently restrictive so that $\pi_\mu < 0$ for all $t \in [0, 1)$, the prices under the quota are higher than what they are under free trade and increasing over time:

$$P^*(t) > P^s(t),\ p^*(t) > p^s(t),\ P^{*\prime}(t) > 0, \text{ and } p^{*\prime}(t) > 0 \text{ for } t \in [0, 1).$$

5. Quotas as facilitating practices?

Krishna (1989) showed for a differentiated-good Bertrand game that both the foreign and the home firm earn greater profits under a quota than under free trade. This result holds when there is a tariff; i.e., a quota results in a more collusive equilibrium than the equal-import tariff. In this section we examine whether this property of the quota holds in the present setting.

As before, focus on the interval [0,1), and set the quota equal to the quantity of imports under free trade:

$$\bar{q}^f = \int_0^1 x(P^s, p^s)dt. \tag{13}$$

The free-trade prices P^s and p^s obviously meet the constraint (13). They fail to satisfy the first-order conditions (9) and (12), however, and hence no longer constitute a subgame-perfect equilibrium.[15] Thus, the imposition of the quota at the free-trade level of imports affects the equilibrium behavior in our model.

Our analysis of section 4 is fully applicable here. Thus, under the quota the domestic firm's best-response function is located outside its free-trade position and shifts out over time. In the normal case, π_μ is negative, so the foreign firm's best-response function behaves similarly. With both the best-response functions being displaced outward from their respective free-trade positions, the equilibrium prices are higher under the quota than under free trade. Thus, $P^{*f}(t) > P^s$ and $p^{*f}(t) > p^s$ for all $t \in [0, 1)$. Then, both firms' profits are greater under the quota $\bar{q}^f$ than under free trade. In other words, a quota facilitates collusion relative to an equal-import tariff.

As mentioned already, however, when the quota is set equal to the volume of imports under free trade, π_μ may be positive at least early in the year, and hence we cannot rule out the possibility that the foreign firm's best-response function shifts in so that the equilibrium prices fall below those under free trade during that period. Then, we cannot say for sure that the quota $\bar{q}^f$ always benefits the domestic firm. However, we can claim the following result.

Proposition 6. Under the quota set at the free-trade level of imports, the foreign firm's profit always exceeds what it is under free trade.

The reason is as follows. Suppose that the foreign firm charges the free-trade price p^s throughout the year under the quota $\bar{q}^f$. Since the domestic firm's best-response function lies farther out from its free-trade position at time zero and keeps moving out, the rates of exports must exceed x^s, meaning that the foreign firm will sell out the total quantity $\bar{q}^f$ before the year ends. Therefore, as long as the interest rate is positive, the foreign firm must make a greater profit per year than under free trade. While it is feasible to charge p^s for the entire quantity $\bar{q}^f$, it is clearly a suboptimal strategy. Pursuing the

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[15] To see this, suppose that firms charge P^s, p^s at time zero (the moment the quota is imposed). Then the

optimal pricing strategy given by the first-order condition (12) therefore must raise the foreign firm's profit even more.

In conclusion, in the normal case with π_{μ} negative at t = 0, a quota raises profits to both firms relative to the equal-import tariff. When q quota is not very restrictive so that π_{μ} is positive for some time early in the year, a quota may result in a smaller total profit to the domestic firm than the equal-import tariff. In that case, the domestic firm may prefer a low-rate tariff to the equal-import and not very restrictive quota.

6. Concluding remarks

This paper explores the fundamental difference between the capacity-constrained firm and the quota-constrained firm. Quotas limit the total quantity the foreign firm is allowed to export within a given period, say, a year. As a consequence the firm under the quota can strategically vary output levels at every instant as long as the sum of exports does not exceed the quota. In contrast, capacity constraints are more physical, limiting the firm's production ability per unit of time, say, due to the plant size. As a result, the capacity-constrained firm does not face the isoperimetric problem as the firm does under the quota. This difference is obscured in the literature examining the relative effect of quota, which analyzes firms under a quota as capacity-constrained ones.

One benefit from treating the quota as an isoperimetric problem is that firms play pure strategies at all times. This fact is helpful in solving the dynamic profit-

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first expression on the right-hand side of (9) vanishes. Since $\Pi_{\mu}[0, \mu(0)]x_P > 0$ the first-order condition (9) maximization problems under the quota. We develop techniques to solve this type of differential games, and fully characterize the equilibrium path for the quota-constrained Bertrand duopoly model, the one-shot version of which was considered in Krishna (1989). The techniques we developed are useful in analyzing other models of one-shot games under the quota that have mixed-strategy equilibriums, e.g., Reitzes and Grawe (1994). We leave it for the future.

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is not satisfied.

Appendix: Proofs of the three lemmas mentioned in the text are presented.

Lemma 1. $(\Pi_{\mu t} + \Pi_{\mu\mu}x)x_P > 0$

Proof: First, since goods are substitutes, we have $x_P > 0$. Therefore, the proof is complete if we show $\Pi_{\mu t} + \Pi_{\mu\mu}x > 0$. The definition of the maximum profit function and the principle of optimality imply

$$\Pi[t, \mu(t)] \equiv \Delta t\{\tilde{P}X(\tilde{P}, \tilde{p}) - C[X(\tilde{P}, \tilde{p})]\}e^{-rt} + \Pi[t + \Delta t, \mu(t) + \Delta\mu]$$

$$= \Delta t\{\tilde{P}X(\tilde{P}, \tilde{p}) - C[X(\tilde{P}, \tilde{p})]\}e^{-rt}$$

$$+ \{\Pi[t, \mu(t)] + \Pi_t[t, \mu(t)]\Delta t + \Pi_\mu[t, \mu(t)]\Delta\mu + \text{h.o.t.}\}$$

where the equality follows from a Taylor series expansion. Dividing through by Δt and taking the limit $\Delta t \to 0$ yields the Bellman equation:

(A1) $$-\Pi_t[t, \mu(t)] = e^{-rt}\{\tilde{P}X(\tilde{P}, \tilde{p}) - C[X(\tilde{P}, \tilde{p})]\} + \Pi_\mu[t, \mu(t)]x(\tilde{P}, \tilde{p}).$$

Since this holds for all μ, we can differentiate (A1) with respect to μ to obtain:

$$-\Pi_{t\mu}[t, \mu(t)] = e^{-rt}\{\tilde{P}X_p(\tilde{P}, \tilde{p}) - C'[X(\tilde{P}, \tilde{p})]X_p\}\tilde{p}_\mu$$

$$+ \Pi_{\mu\mu}[t, \mu(t)]x(\tilde{P}, \tilde{p}) + \Pi_\mu[t, \mu(t)]x_p(\tilde{P}, \tilde{p})\tilde{p}_\mu,$$

which simplifies to

(A2) $$\Pi_{\mu\mu}[t, \mu(t)]x(\tilde{P},\tilde{p}) + \Pi_{t\mu}[t, \mu(t)]$$

$$= -\{e^{-rt}\{\tilde{P}X_p(\tilde{P}, \tilde{p}) - C'[X(\tilde{P}, \tilde{p})]X_p\} + \Pi_\mu[t, \mu(t)]x_p(\tilde{P}, \tilde{p})\}\tilde{p}_\mu.$$

Rearranging the first-order condition (9) yields

$$e^{-rt}\{\tilde{P}X_P(\tilde{P}, \tilde{p}) - C'[X(\tilde{P}, \tilde{p})]X_P(\tilde{P}, \tilde{p})\}$$

$$= - \Pi_\mu[t, \mu(t)]x_P(\tilde{P}, \tilde{p}) - e^{-rt}X(\tilde{P}, \tilde{p}).$$

Substituting this expression to the right-hand side of (A2), we obtain

$$\Pi_{\mu\mu}[t, \mu(t)]x(\tilde{P}, \tilde{p}) + \Pi_{t\mu}[t, \mu(t)]$$

$$= - \{e^{-rt}\{\tilde{P}X_p(\tilde{P}, \tilde{p}) - C'[X(\tilde{P}, \tilde{p})]X_p\} + \Pi_\mu[t, \mu(t)]x_p(\tilde{P}, \tilde{p})\}\tilde{p}_\mu$$

$$= - \{ - e^{-rt}X(P, p) + \Pi_\mu[t, \mu(t)][x_p(\tilde{P}, \tilde{p}) - x_P(\tilde{P}, \tilde{p})]\}\tilde{p}_\mu$$

Since we have $\Pi_\mu > 0$, the terms in the braces are negative. Also, in Lemma 3 below, we show $\tilde{p}_\mu > 0$. Therefore, we have $\Pi_{\mu\mu}x + \Pi_{t\mu} > 0$. o

Lemma 2. $(\pi_{\mu t} + \pi_{\mu\mu}x)x_p > 0$

Proof: Following a procedure similar to the one leading to (A1) in the proof of Lemma 1, we can obtain the Bellman equation for the foreign firm:

(A3) $$- \pi_t[t, \mu(t)] = e^{-rt}\{\tilde{p}x(\tilde{P}, \tilde{p}) - c[x(\tilde{P}, \tilde{p})]\} + \pi_\mu[t, \mu(t)]x(\tilde{P}, \tilde{p}).$$

Differentiating (A3) with respect to μ yields:

$$- \pi_{t\mu}[t, \mu(t)] = e^{-rt}\{\tilde{p} - c'[x(\tilde{P}, \tilde{p})]\}x_P(\tilde{P}, \tilde{p})\tilde{P}_\mu$$

$$+ \pi_{\mu\mu}[t, \mu(t)]x(\tilde{P}, \tilde{p}) + \pi_\mu[t, \mu(t)]x_P(\tilde{P}, \tilde{p})\tilde{P}_\mu$$

or

(A4) $\pi_{t\mu}[t, \mu(t)] + \pi_{\mu\mu}[t, \mu(t)]x(\widetilde{P}, \widetilde{p})$

$$= - \{e^{-rt}\{\widetilde{p} - c'[x(\widetilde{P}, \widetilde{p})]\} + \pi_{\mu}[t, \mu(t)]\}x_P(\widetilde{P}, \widetilde{p})\widetilde{P}_{\mu}.$$

The expression in braces on the right-hand side of (A4) is positive since rewriting the first-order condition (12) from the text yields:

$$e^{-rt}\{\widetilde{p} - c'[x(\widetilde{P}, \widetilde{p})]\} + \pi_{\mu}[t, \mu(t)] = - e^{-rt}x(\widetilde{P}, \widetilde{p})/x_p(\widetilde{P}, \widetilde{p}) > 0.$$

Furthermore, $x_P > 0$ because products are substitutes, and $\widetilde{P}_{\mu} > 0$ as we prove later in Lemma 3. We have shown that the right-hand side of (A4) is negative. Lemma 2 then follows immediately from the fact that $x_p < 0$. o

Lemma 3. $\widetilde{p}_{\mu} > 0$ and $\widetilde{P}_{\mu} > 0$.

Proof: Differentiating the first-order conditions (9) and (12) simultaneously yields

(A5) $|D|dP/d\mu = - \Pi_{\mu\mu}x_P\psi_p + \pi_{\mu\mu}x_p\Psi_p$

$|D|dp/d\mu = \Pi_{\mu\mu}x_P\psi_P - \pi_{\mu\mu}x_p\Psi_P$

where $|D| \equiv \psi_p\Psi_P - \psi_P\Psi_p > 0$ by the standard stability and uniqueness conditions. We also know that $\Psi_P < 0$, $\psi_p < 0$ by the second-order conditions, and $\Psi_p > 0$, $\psi_P > 0$ due to strategic complementarity of prices.

We first show that $\pi_{\mu\mu} < 0$. To do so, note that

(A6) $\pi(t, \mu_0)$

$$> \pi(t, \mu_1) + (\mu_1 - \mu_0)e^{-rt}(x + px_p - c'x_p)/x_p$$

$$= \pi(t, \mu_1) - (\mu_1 - \mu_0)\pi_\mu(t, \mu_0)$$

for given t, μ_1, and μ_0 ($\mu_1 > \mu_0$). The inequality holds for the following reason. Both sides of the inequality measure profits the foreign firm makes by exporting the quantity $\bar{q} - \mu_0$ between time t and 1. The left-hand side, $\pi(t, \mu_0)$, is by definition the foreign firm's maximum profit attainable. The first term, $\pi(t, \mu_1)$, on the right-hand side of the inequality is the maximum profit attainable by exporting the smaller quantity $\bar{q} - \mu_1$ during this period. The next term represents the profit resulting from exporting the quantity, $\mu_1 - \mu_0$, at $e^{-rt}(x + px_p - c'x_p)$. However, since this term is not the maximum profit the foreign firm can make by exporting $\mu_1 - \mu_0$ units, the inequality is established. The equality in (A6) follows from the first-order condition (12). But (A6) means that $\pi(t, \mu)$ is concave in μ: that is, $\pi_{\mu\mu} < 0$.

We next show that $\Pi_{\mu\mu} > 0$. Let μ_1 and μ_0 be given, with $\mu_1 > \mu_0$. $\Pi(t, \mu_1)$ represents the maximum profit the domestic firm can make when $\bar{q} - \mu_1$ units of the foreign good are imported between t and 1. If $\bar{q} - \mu_0$ units were allowed to be imported instead during the same period, the domestic firm would suffer the loss $\Pi(t, \mu_1) - \Pi(t, \mu_0) > 0$. Now suppose that the domestic firm receives a compensation for this loss in the amount of

$$- e^{-rt}(X + PX_P - C'X_P)(\mu_1 - \mu_0)/x_P.$$

This has the following interpretation. Since x_P measures the increase in imports induced by a unit increase in the domestic price, $(\mu_1 - \mu_0)/x_P$ represents the price increase necessary to induce the increase in imports by $\mu_1 - \mu_0$. But according to the first-order condition (9), the marginal cost of a price increase (a deviation from optimality) is

$$- e^{-rt}(X + PX_P - C'X_P).$$

However, since the profit function is concave, this compensation evaluated at the margin underestimates the true cost of the discrete price change necessary to induce the increase in imports, $\mu_1 - \mu_0$. Therefore,

$$\Pi(t, \mu_1) - \Pi(t, \mu_0) > - (\mu_1 - \mu_0)e^{-rt}(X + PX_P - C'X_P)/x_P$$

$$= (\mu_1 - \mu_0)\Pi_\mu(t, \mu_0),$$

where the equality follows from (9). However, this means that $\Pi(t, \mu)$ is convex in μ, and hence $\Pi_{\mu\mu} > 0$. Using these results in (A5), we have $\tilde{p}_\mu > 0$ and $\tilde{P}_\mu > 0$. o

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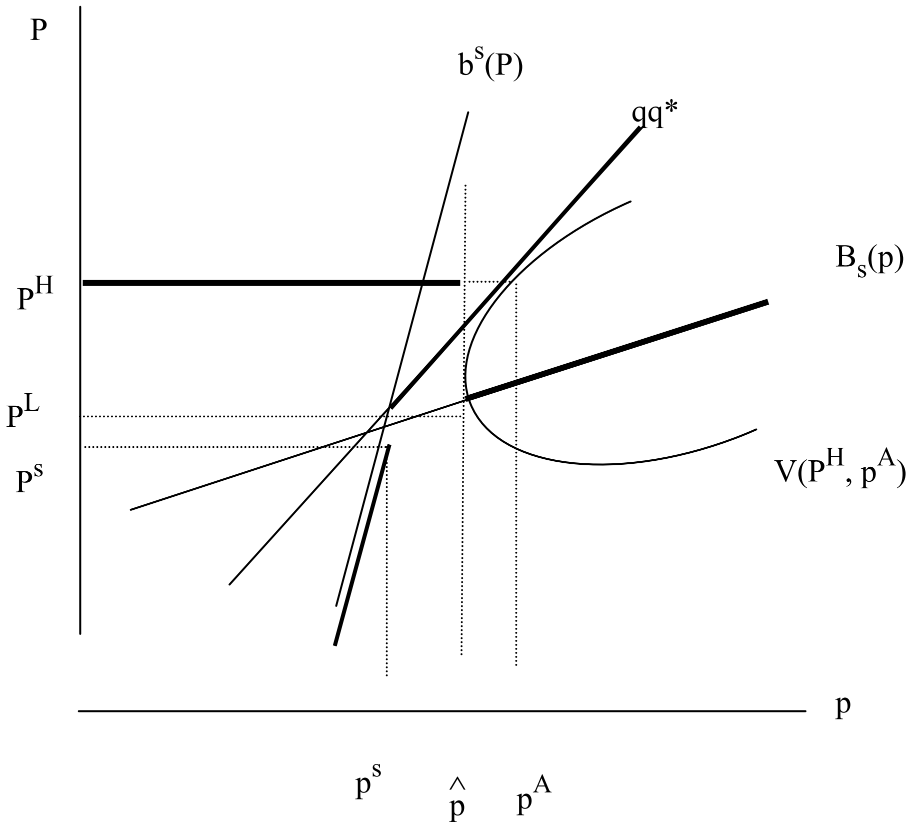

Figure 1: Equilibrium under free trade and under the quota