Supplementary material for “Nonparametric LAD cointegrating regression”

Details of (23), (25), (27), (33), and (34) are given here.

(23), (25), (27):
We can establish (23), (25), and 27) by combining the standard arguments in the literature of nonparametric quantile regression. First put

\[ a_i = \frac{1}{2} \left( \frac{X_i - x_0}{h} \right)^2 h^2 g''(\bar{X}_i) \quad \text{and} \quad b_i = \tau_n^{-1} \eta_i^T \theta. \]

and notice that \( a_i \) and \( b_i \) tends to 0 uniformly in \( i \) since we can assume \( |X_i - x_0| \leq Ch \).

(23): \( v_i^* \) is defined in (9) as \( v_i^* = v_i + a_i \). Then

\[ |v_i^*| \leq C|b_i| \Rightarrow -C|b_i| - a_i \leq v_i \leq C|b_i| - a_i. \]

Recall that \( a_i/\tau_n^{-1} = O(1) \) uniformly in \( i \) from Assumption H. Hence we obtain (23) from from Assumptions V and U2.

(25): When \( a_i \geq 0 \) and \( b_i \geq 0 \), \( B_{2i}(\theta) \) is not 0 only when \(-a_i \leq v_i \leq -a_i + b_i\). Then we have

\[ B_{2i}(\theta) = -2(v_i + a_i - b_i) \]

and

\[ -2 \int_{-a_i}^{v_i + a_i - b_i} (v_i + a_i - b_i)f_{v_i}(v_i|\mathcal{E})dv_i = b_i^2 f_{v_i}(0|\mathcal{E}) + o_p(b_i^2) \]

uniformly in \( i \) from Assumption V and U2. We can deal with the other cases in the same way.

(27): When \( a_i > 0 \), we have

\[ \text{sign}(v_i^*) - \text{sign}(v_i) = 2I(-a_i < v_i < 0) \]

and

\[ 2(F_{v_i}(0|\mathcal{E}) - F_{v_i}(-a_i|\mathcal{E})) = 2a_i f_{v_i}(0|\mathcal{E}) + o_p(a_i) \]
uniformly in $i$ from Assumptions V and U2. We can deal with the other case in the same way.

(33), (34):

It is not easy to establish (33) and (34).

(33): Recall that $v_i^{**} = v_i + \delta_i^{**}$ and $\delta_i^{**} = O(h)$.

When $\delta_i^{**} > 0$, we have

$$\text{sign}(v_i^{**}) - \text{sign}(v_i) = 2I(-\delta_i^{**} < v_i < 0)$$

and

$$2h^{-2}(F_{v_i}(0|\mathcal{E}) - F_{v_i}(-\delta_i^{**}|\mathcal{E}))$$

$$= 2h^{-2}\delta_i^{**}f_{v_i}(0|\mathcal{E}) - h^{-2}(\delta_i^{**})^2 f_{v_i}'(0|\mathcal{E}) + O(|f_{v_i}'(0|\mathcal{E}) - f_{v_i}'(\tilde{\delta}_i^{**}|\mathcal{E})|),$$

where $\tilde{\delta}_i^{**}$ is between 0 and $\delta_i^{**}$.

Assumption V and (35) imply that

$$|f_{v_i}'(0|\mathcal{E}) - f_{v_i}'(\tilde{\delta}_i^{**}|\mathcal{E})| \leq C(\sup_{|m_u - m_i| \leq Ch} |f_{u_i}'(m_u|\mathcal{E}_i^{i-m_0}) - f_{u_i}'(m|\mathcal{E}_i^{i-m_0})| + o(1))$$

uniformly in $i$. Since

$$\lim_{h \to 0} E\{\sup_{|m_u - m_i| \leq Ch} |f_{u_i}'(m_u|\mathcal{E}_i^{i-m_0}) - f_{u_i}'(m|\mathcal{E}_i^{i-m_0})|\} = 0,$$

we get from [22] (for example, see Proposition 1 of this paper) that

$$\tau_n^{-2} \sum_{i=1}^n K_i |f_{v_i}'(0|\mathcal{E}) - f_{v_i}'(\tilde{\delta}_i^{**}|\mathcal{E})| = o_p(1).$$

(34): First write

$$f_{v_i}(0|\mathcal{E}) = f_{u_i}(m_u|\mathcal{E}_i^{i-m_0})(\frac{\partial w}{\partial u}(X_i, m_u))^{-1}$$

$$= f_{u_i}(m_u|\mathcal{E}_i^{i-m_0})(w(x_0) + (X_i - x_0)w'(x_0) + o(|X_i - x_0|)).$$
where \( w(x) \) is clearly defined in the above equation. Using the above notation, we have

\[
\frac{2h^{-2}}{\tau_n^2} \sum_{i=1}^{n} K_i \delta_i^f \{ f_i(0) | \mathcal{E} \} \\
= \frac{2h^{-2}}{\tau_n^2} \sum_{i=1}^{n} K_i \left\{ \left( \frac{X_i - x_0}{h} \right) h g'(x_0) + \frac{1}{2} \left( \frac{X_i - x_0}{h} \right)^2 h^2 g''(\bar{X}) \right\} \\
\times f_u(m_u | \mathcal{E}_{i-m_0}^i)(w(x_0) + (X_i - x_0)w'(x_0) + o(|X_i - x_0|)) \\
= 2\tau_n^{-2} \sum_{i=1}^{n} K_i f_u(m_u | \mathcal{E}_{i-m_0}^i) \left\{ \left( \frac{X_i - x_0}{h} \right)^2 g'(x_0)w'(x_0) \right\} \\
+ \frac{1}{2} \left( \frac{X_i - x_0}{h} \right)^2 g''(x_0)w(x_0) + o(h^2) \}
\]

(37)

We can handle the first term of (37) by using Proposition 1.

Finally we consider the second term of (37). Write

\[
\frac{h^{-1}}{\tau_n^2} \sum_{i=1}^{n} K_i \frac{X_i - x_0}{h} f_u(m_u | \mathcal{E}_{i-m_0}^i) \\
= \frac{\tau_n^{-1}}{(nh^6)^{1/4}} \sum_{i=1}^{n} K_i \frac{X_i - x_0}{h} \left\{ (f_u(m_u | \mathcal{E}_{i-m_0}^i) - f_u(m_u)) + f_u(m_u) \right\}. \quad (38)
\]

We can use Theorem 2.1 of [23] to show \( \tau_n^{-1} \sum_{i=1}^{n} (X_i - x_0)/h) K_i = O_p(1) \). We can deal with the first term inside the braces of (38) by using a result similar to the second element of Proposition 1.