

# Persistent catastrophic shocks and equity premiums: A note\*

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## CATASTROPHIC SHOCKS AND EQUITY PREMIUMS

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## **Abstract**

This note demonstrates analytically that a persistent catastrophic shock on endowment growth, even if moderate, yields negative equity premiums when a representative agent is relatively prudent. In particular, it derives the minimum persistence necessary to have zero equity premiums.

Key words: catastrophic shocks, persistence, equity premiums

# 1 Introduction

Several papers, including Rietz (1988) and Barro (2006), claim that a catastrophic shock on endowment growth generates large equity premiums in an exchange economy with a representative agent. However, the following intuitive argument anticipates that this claim may not hold in cases in which shocks are persistent. When the realized catastrophic shock lowers endowment growth substantially, equity prices may soar due to strong precautionary savings by prudent consumers against the possibility of another catastrophic shock. Consequently, such a negative correlation between consumption growth and equity returns may generate negative equity premiums.

As pointed out by Campbell (1999) and others, the same logic is applicable to the case in which positively correlated productivity growth shocks work to reduce equity premiums. However, when shocks on endowment growth are catastrophic, persistent shocks may yield not only lower, but also negative equity premiums. Barro et al. (2009) provide numerical examples, thereby showing that persistent catastrophic shocks lead to negative equity premiums under empirically plausible setups.

This note analytically confirms the above conjecture in an exchange economy of a representative agent with constant relative risk aversion.<sup>1</sup> In particular, we derive precisely the minimum persistence necessary to have zero equity premiums.

## 2 Setup

### 2.1 A simple exchange economy with a representative agent

Following Mehra and Prescott (1985) and Rietz (1988), we construct a single goods exchange economy with a representative agent. There are two financial assets, risk-free assets and Lucas trees, which are assigned claims on stochastic endowment. In this paper, the Lucas tree and equity are interchangeable.

Here,  $p_t^f$  and  $p_t^e$  denote the time- $t$  prices of risk-free assets and Lucas trees, respectively.  $Y_t$  is dividends on Lucas trees at time  $t$ , and is regarded as endowment.  $f_t$  and  $e_t$  indicate the outstanding positions of the two financial assets. A representative agent

has a time-additive utility  $u(C)$  on consumption at time  $t$  ( $C_t$ ) with a discount factor  $\beta$ .

Given the above setup, the representative agent maximizes the following lifetime utility:

$$\max E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(C_t) \right],$$

subject to  $C_t = Y_t e_{t-1} + p_t^e (e_{t-1} - e_t) + f_{t-1} - p_t^f f_t$ , where  $E_0$  denotes the expectation operator conditional on time-0 information.

The first-order conditions or Euler equations are

$$p_t^e = E_t \left[ \beta \frac{u'(C_{t+1})}{u'(C_t)} \{ Y_{t+1} + p_{t+1}^e \} \right], \quad (1)$$

$$p_t^f = E_t \left[ \beta \frac{u'(C_{t+1})}{u'(C_t)} \right]. \quad (2)$$

The market clearing conditions consist of  $C_t = Y_t$ ,  $e_t = 1$ , and  $f_t = 0 \forall t$ . To obtain concrete analytical results, we assume below a preference with constant relative risk aversion (CRRA), or  $u(C_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma}$  with  $0 < \gamma$ , where  $\gamma$  indicates the degree of relative risk aversion.

We now characterize persistent catastrophic shocks on endowment growth.<sup>2</sup> The aggregate endowment or consumption grows according to  $Y_{t+1} = x_{t+1} Y_t$  or  $C_{t+1} = x_{t+1} C_t$ . A gross growth rate  $x_{t+1}$  follows the Markov process:  $\text{Prob}(x_{t+1} = x_j | x_t = x_i) = q_{i,j}$ . There are two states, a normal state ( $n$ ) and a catastrophic state ( $c$ ). Here,  $x_c \equiv x_n - \kappa$ , where  $\kappa (> 0)$  denotes the size of catastrophic growth shocks. In a normal state at time  $t$ ,  $x_{t+1}$  takes  $x_n$  with probability  $q_{n,n} = 1 - \phi$ , and  $x_c$  with probability  $q_{n,c} = \phi$ . In a catastrophic state at time  $t$ ,  $x_{t+1}$  takes  $x_c$  with probability  $q_{c,c} = \psi$ , and  $x_n$  with probability  $q_{c,n} = 1 - \psi$ .

Given the above characterization, endowment growth shocks are named *catastrophic* with rather tiny  $\phi$  and extremely large  $\kappa$ . In addition, a catastrophic growth shock is called *persistent* if  $\phi < \psi$ , and it is regarded as *purely transitory* if  $\phi = \psi$ . The unconditional probability of a catastrophic state or  $\theta$  is equal to  $\frac{\phi}{1-\psi+\phi}$ .

Euler equations (1) and (2) can be rewritten in a recursive manner:

$$p^e(Y_t, x_t) = E_t \left[ \beta \frac{u'(Y_{t+1})}{u'(Y_t)} \left\{ Y_{t+1} + p^e(Y_{t+1}, x_{t+1}) \right\} \right], \quad (3)$$

$$p^f(Y_t, x_t) = E_t \left[ \beta \frac{u'(Y_{t+1})}{u'(Y_t)} \right]. \quad (4)$$

A price dividend ratio  $w_i$  is defined as  $p^e(C, i) = w_i C$ . Then, exploiting the property of CRRA preference, we can further rewrite equation (3) as follows:

$$\mathbf{w} \equiv \begin{bmatrix} w_n \\ w_c \end{bmatrix} = \begin{bmatrix} \beta(1-\phi)u'(x_n)x_n & \beta\phi u'(x_c)x_c \\ \beta(1-\psi)u'(x_n)x_n & \beta\psi u'(x_c)x_c \end{bmatrix} \begin{bmatrix} w_n + 1 \\ w_c + 1 \end{bmatrix} \quad (5)$$

A realized gross return on equity between state  $(C, i)$  and state  $(x_j C, j)$  is

$$r_{ij}^e = \frac{p^e(x_j C, j) + x_j c}{p^e(C, i)} = \frac{x_j(w_j + 1)}{w_i}. \quad (6)$$

In equation (6),  $x_j \frac{w_j}{w_i}$  corresponds to a capital gain, while  $\frac{x_j}{w_i}$  matches an income gain. Accordingly, the expected return on equity in state  $(C, i)$  is

$$R_i^e = \sum_{j=\{n,c\}} q_{ij} r_{ij}^e.$$

A risk-free rate  $R_i^f$  in state  $(C, i)$  is equal to  $\frac{1}{p^f(C, i)}$ . The unconditional expectations of equity returns and risk-free rates are  $R^e = (1-\theta)R_n^e + \theta R_c^e$  and  $R^f = (1-\theta)R_n^f + \theta R_c^f$ . Thus, the unconditional equity premium is defined as  $\Pi \equiv R^e - R^f$ .

## 2.2 The minimum persistence necessary to have negative equity premiums

In this subsection, we derive analytically how small persistence  $\psi$  would yield negative equity premiums. As in a standard time-additive preference, the unconditional equity

premium is

$$\Pi = -\left\{ (1 - \theta) \frac{\text{cov}_n[u'(x), R^e]}{E_n[u'(x)]} + \theta \frac{\text{cov}_c[u'(x), R^e]}{E_c[u'(x)]} \right\}. \quad (7)$$

With CRRA preferences, equation (7) implies that negative equity premiums emerge when equity returns are negatively correlated with consumption growth (or endowment growth in an exchange economy). First, we demonstrate below that persistence ( $\psi > \phi$ ) works to reduce positive correlation between equity returns and consumption growth.

As implied by equation (6), when consumption growth declines by  $1 - x_c$  due to catastrophic shocks, the positive correlation between equity returns and consumption growth is weakened with  $\frac{w_c}{w_n} > 1$ . That is, when price dividend ratios are larger in a catastrophic state than in a normal state, the correlation is less positive.

From equation (5), we derive

$$\mathbf{w} = (\mathbf{I} - \mathbf{M})^{-1} \mathbf{M} \mathbf{i}, \quad (8)$$

where

$$\mathbf{M} \equiv \begin{bmatrix} \beta(1 - \phi)u'(x_n)x_n & \beta\phi u'(x_c)x_c \\ \beta(1 - \psi)u'(x_n)x_n & \beta\psi u'(x_c)x_c \end{bmatrix},$$

$\mathbf{I}$  is the identity matrix, and  $\mathbf{i}$  is the column vector whose elements are ones.

The solution to equation (8) is

$$w_n = \frac{\beta}{D(x_n, \kappa, \phi, \psi)} \left[ \phi u'(x_c)x_c + u'(x_n)x_n \{1 - \phi - \beta(\psi - \phi)u'(x_c)x_c\} \right], \quad (9)$$

$$w_c = \frac{\beta}{D(x_n, \kappa, \phi, \psi)} \left[ \psi u'(x_c)x_c + u'(x_n)x_n \{1 - \psi - \beta(\psi - \phi)u'(x_c)x_c\} \right], \quad (10)$$

where

$$D(x_n, \kappa, \phi, \psi) \equiv 1 - \beta\psi u'(x_c)x_c - \beta u'(x_n)x_n \{1 - \phi - \beta(\psi - \phi)u'(x_c)x_c\}.$$

$D(x_n, \kappa, \phi, \psi)$  needs to be positive for a positive value of equity prices.

From equations (9) and (10), we have

$$w_c - w_n = \frac{\beta}{D(x_n, \kappa, \phi, \psi)} \delta(x_n, \kappa)(\psi - \phi), \quad (11)$$

where

$$\delta(x_n, \kappa) \equiv u'(x_c)x_c - u'(x_n)x_n.$$

$\delta(x_n, \kappa)$  is positive when  $\gamma$  is greater than one. Equation (11) implies that  $\frac{w_c}{w_n}$  is greater than one if  $\gamma$  is larger than one, and  $\psi$  is greater than  $\phi$ . Thus, the correlation between equity returns and consumption growth is less positive when growth shocks are persistent and consumers are prudent. That is, the persistence of growth shocks and the prudence of consumers are necessary for negative equity premiums, but are not sufficient. The following proposition demonstrates that catastrophic growth shocks may lead to negative equity premiums with moderate persistence.

**Proposition 1** *If there exists a negative equity premium ( $\Pi < 0$ ), then  $\psi$  satisfies  $\phi < \psi_0 < \psi < 1$ , where  $\psi_0$  is defined as:*

$$\psi_0 \equiv \phi + \frac{\kappa}{\beta x_n x_c \{u'(x_c) - u'(x_n)\}}. \quad (12)$$

See Appendix 1 for proof.

Proposition 1 demonstrates that mild persistence of catastrophic shocks and the prudence of consumers are required to have negative equity premiums in the following respects. First, tiny  $\phi$  characteristic of catastrophic shocks itself reduces the minimum persistence  $\psi_0$  defined by equation (12).

Second, large  $\kappa$  also characteristic of catastrophic shocks lowers  $\psi_0$ . As shown in Appendix 2, by Taylor-expanding equation (12) with respect to  $\kappa$  in the neighborhood of  $x_c = x_n$  up to second order, we can prove that  $\frac{\partial \psi_0}{\partial \kappa} < 0$  and  $\frac{\partial^2 \psi_0}{\partial \kappa^2} > 0$  if  $\gamma > 1$ . That

is,  $\psi_0$  is decreasing and convex with respect to  $\kappa$ .

Third, equation (12) indicates that more prudence or larger  $\gamma$  also contributes to reducing the minimum persistence  $\psi_0$ ; the denominator of the second term in its right-hand side  $\{u'(x_c) - u'(x_n)\}$  becomes larger with more sizable  $\gamma$ . Conversely, stronger persistence is required with less prudent consumers. For example, when a preference is logarithmic ( $\gamma = 1$ ),  $\psi_0 = \phi + \frac{1}{\beta}$  at the limit of  $\kappa = 0$ . Accordingly, it is impossible to have negative equity premiums with  $\beta < 1$  because  $\psi_0$  becomes larger than one.

One caveat concerning Proposition 1 is that  $\psi_0$  is not sufficient, but is necessary to have zero equity premiums. Depending on a set of structural parameters, equilibrium may not exist at  $\psi = \psi_0$ .

### 2.3 Calibration

This subsection presents some results of simple calibration. Following Barro (2006), we assume that  $\beta = 0.97$ ,  $\gamma = 4$ ,  $x_n = 1.025$ , and  $\phi = 0.017$ . Next, we set  $\kappa$  equal to 0.4 such that the unconditional equity premium ( $\Pi$ ) may match the historical average (4% per year). Under this assumption,  $\psi_0 = 0.131$ . Then, a persistence parameter  $\psi$  is set at either 0.017 (purely transitory), 0.070, or 0.135 ( $> \psi_0 = 0.131$ ).

Figure 1 plots the unconditional equity premium given a persistence parameter  $\psi$ . As this figure shows, the unconditional equity premium declines quickly as catastrophic shocks are persistent. That is, large equity premiums disappear completely with mild persistence.

[Figure 1]

Table 1 reports some results of the above calibration exercise. For comparison, this table also includes the case of a logarithmic preference ( $\gamma = 1$ ). When  $\gamma = 4$ , the unconditional equity premium reduces from 0.04 ( $\psi = 0.017$ , or a purely transitory case) to 0.027 if  $\psi = 0.070$ , and to  $-0.003$  if  $\psi = 0.135$ .

A major reason for such substantial decreases in equity premiums is that the correlation between equity returns and consumption growth becomes negative as catastrophic

shocks are relatively persistent. When catastrophic shocks are purely transitory, price dividend ratios do not differ between  $w_n$  and  $w_c$ , and negative (net) equity returns ( $0.656 - 1.000 = -0.344$ ) are realized upon the occurrence of catastrophic shocks. On the other hand, a net equity return increases from  $-0.344$  to  $-0.220$  if  $\psi = 0.070$ , and it becomes even positive ( $0.059$ ) if  $\psi = 0.135$ .

If  $\gamma$  increases from four, then the size of equity premiums ( $\Pi$ ) is more sensitive to a persistence parameter  $\psi$ . We again set  $\kappa$  to have 4% per year for  $\Pi$  given  $\gamma$ ;  $\kappa = 0.315$  for  $\gamma = 6$ ,  $\kappa = 0.263$  for  $\gamma = 8$ ,  $\kappa = 0.227$  for  $\gamma = 10$ , and  $\kappa = 0.171$  for  $\gamma = 15$ . Then, we compute the minimum persistence  $\psi_0$  for each case;  $\psi_0 = 0.081$  for  $\gamma = 6$ ,  $\psi_0 = 0.061$  for  $\gamma = 8$ ,  $\psi_0 = 0.050$  for  $\gamma = 10$ , and  $\psi_0 = 0.037$  for  $\gamma = 15$ . In all of these cases, equilibrium exists at  $\psi = \psi_0$ . Among more prudent consumers, large equity premiums disappear more quickly under less persistence ( $\psi_0$  closer to  $\phi = 0.017$ ).

The case with a logarithmic preference ( $\gamma = 1$ ) offers a contrasting result. The unconditional equity premium is quite small, and increases slightly with the persistence of catastrophic shocks; it is 0.004 in a purely transitory case, and it is 0.006 with  $\psi = 0.135$ . A reason for this consequence is that the correlation between equity returns and consumption growth is always positive in the case of a logarithmic preference.

[Table 1]

### 3 Discussion

This note has derived analytically the minimum persistence of catastrophic shocks on endowment growth to have zero equity premiums under CRRA preferences. In addition, it has demonstrated numerically that under a reasonable set of structural parameters, large equity premiums quickly disappear when catastrophic shocks are mildly persistent.

As demonstrated in Maddison (2003), Barro (2006), Barro et al. (2009), and others, catastrophic shocks on economic growth are often observed to be persistent. Given the extreme sensitivity of equity premiums to the persistence of catastrophic shocks, we should be deeply cautious of attributing the emergence of large equity premiums to the

presence of catastrophic shocks.

## Notes

<sup>1</sup>Barro et al. (2009) suggest that this conjecture may not hold under nonexpected preferences with elastic intertemporal substitution. On the other hand, using the same class of nonexpected preferences, Gourio (2008) shows that equity premiums become not negative, but rather low given quick recoveries in the aftermath.

<sup>2</sup>Our formulation of catastrophic shocks is rather different from that of Cecchetti, Lam, and Mark (1990), who adopt a Markov regime switching model to characterize the normal and depression regimes. In their formulation, a regime, either of the two regimes, is fixed prior to the realization of aggregate shocks.

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## Appendix

### Appendix 1: Proof of Proposition 2

As equation (7) implies, the unconditional equity premiums are negative when both  $\text{cov}_n[u'(x), R^e]$  and  $\text{cov}_c[u'(x), R^e]$  are positive. After some manipulation, we have

$$\text{cov}_n[u'(x), R^e] = \left[ (1 - \phi) \phi \left\{ u'(x_c) - u'(x_n) \right\} \left\{ r_{n,c}^e - r_{n,n}^e \right\} \right].$$

Because  $u'(x_c) > u'(x_n)$ ,  $r_{n,c}^e - r_{n,n}^e > 0$  implies  $\text{cov}_n[\beta u'(x), R^e] > 0$ .

$r_{n,c}^e - r_{n,n}^e > 0$  is rewritten as  $r_{n,c}^e - r_{n,n}^e = (x - \kappa) \frac{w_c + 1}{w_n} - x \frac{w_n + 1}{w_n}$ . Substituting equations (9) and (10), we have

$$\psi > \phi + \frac{\kappa}{\beta x_n x_c \{ u'(x_c) - u'(x_n) \}}.$$

Accordingly,  $\psi_0 \equiv \phi + \frac{\kappa}{\beta x_n x_c \{ u'(x_c) - u'(x_n) \}}$ .

Similarly,  $\text{cov}_c[\beta u'(x_n), R^e]$  implies  $\psi_0 \equiv \phi + \frac{\kappa}{\beta x_n x_c \{ u'(x_c) - u'(x_n) \}}$ .

(Q.E.D.)

### Appendix 2: Derivation of first and second derivatives of equation (12)

Equation (12) is rewritten as

$$\psi_0 = \phi + \frac{\kappa}{\beta x_n \delta(x_n, \kappa) + \beta u'(x_n) x_n \kappa},$$

where  $\delta(x_n, \kappa) \equiv u'(x_c) x_c - u'(x_n) x_n$ .

Taylor-expanding  $\delta(x_n, \kappa)$  with respect to  $\kappa$  at the neighborhood of  $x_c = x_n$  up to

second order leads to

$$\delta(x_n, \kappa) \simeq u'(x_n)\{\gamma - 1\}\kappa - \frac{u''(x_n)}{2}\{\xi - 2\}\kappa^2,$$

where  $\gamma \equiv -\frac{u''(x_n)}{u'(x_n)}x_n$  (relative risk aversion), and  $\xi \equiv -\frac{u'''(x_n)}{u''(x_n)}x_n$  (relative prudence). Note that  $\xi = \gamma + 1$  under CRRA preference. Then,  $\beta x_n \delta(x_n, \kappa) + \beta u'(x_n)x_n \kappa$  can be rewritten as  $\beta u'(x_n)x_n \gamma \kappa - \beta \frac{u''(x_n)x_n}{2}\{\xi - 2\}\kappa^2$ .

Given the above preparation, we have

$$\psi_0 = \phi + \frac{1}{g\kappa + h}$$

where  $g = -\frac{1}{2}\beta u''(x_n)x_n(\xi - 2)$ , and  $h = \beta u'(x_n)x_n \gamma$ .

Then, we have the following derivatives with respect to  $\kappa$ :

$$\frac{\partial \psi_0}{\partial \kappa} = -\frac{g}{(g\kappa + h)^2} = \frac{\beta u''(x_n)x_n(\xi - 2)}{2[\beta x_n(x_n - \kappa)\{u'(x_c) - u'(x_n)\}]^2} < 0,$$

and

$$\frac{\partial^2 \psi_0}{\partial \kappa^2} = \frac{2g^2}{(g\kappa + h)^3} = \frac{[\beta u''(x_n)x_n(\xi - 2)]^2}{2[\beta x_n(x_n - \kappa)\{u'(x_c) - u'(x_n)\}]^3} > 0.$$

Table 1: Calibration results ( $\beta = 0.97$ ,  $x_n = 1.025$ , and  $\phi = 0.017$ )

	(1)	(2)	(3)	(4)	(5)	(6)
$\gamma$	1	1	1	4	4	4
$\kappa$	0.4	0.4	0.4	0.4	0.4	0.4
$\psi$	0.017	0.07	0.135	0.017	0.07	0.135
$\Pi$	0.004	0.005	0.005	0.040	0.027	-0.003
$R^e$	1.050	1.049	1.049	1.068	1.052	1.018
$R^f$	1.045	1.045	1.044	1.029	1.025	1.021
$w$	32.3	32.3	32.3	20.3	29.3	1090.8
$w_n$	32.3	32.3	32.3	20.3	29.2	1076.6
$w_c$	32.3	32.3	32.3	20.3	35.4	1812.1
$r_{nn}^e$	1.057	1.057	1.057	1.076	1.060	1.026
$r_{nc}^e$	0.644	0.644	0.644	0.656	0.780	1.053
$Ex$	1.018	1.018	1.017	1.018	1.018	1.017

$\gamma$ : relative risk aversion;  $\psi$ : persistence of catastrophic shocks;  $\Pi$ : unconditional equity premiums;  $R^e$ : the unconditional expectation of equity returns;  $R^f$ : the unconditional expectation of risk-free rates;  $w$ : the unconditional expectation of price dividend ratios;  $w_i$ : the price dividend ratio conditional on either a normal state ( $i = n$ ) or a catastrophic state ( $i = c$ );  $r_{ni}^e$ : the realized gross equity return conditional on either of the two states; and  $Ex$ : the unconditional expectation of consumption growth.

Figure 1: The relationship between the unconditional equity premium ( $\Pi$ ) and persistence parameters ( $\psi$ ) ( $\beta = 0.97$ ,  $\gamma = 4$ ,  $\kappa = 0.4$ ,  $x_n = 1.025$ , and  $\phi = 0.017$ )

