ROBUST UTILITY MAXIMIZATION WITH RANDOM ENDOWMENT AND VALUATION OF CONTINGENT CLAIMS UNDER MODEL UNCERTAINTY

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1. OVERVIEW

The goal of this thesis is to develop a *convex duality theory* for robust utility maximization with *unbounded random endowment*. Suppose we are given a semimartingale S, a utility function U, a set of admissible integrands (strategies) Θ, a set of probability measures $\mathcal{P}$ and a random variable B. Then the problem is to:

$$\text{(1.1)} \qquad \text{maximize} \quad \inf_{P\in\mathcal{P}} E^P[U(\theta\cdot S_T + B)], \quad \text{among } \theta\in\Theta.$$

The set $\mathcal{P}$ describes the so-called *model uncertainty* (also called *Knightian uncertainty* in Economics), while the random variable (endowment) B expresses the terminal payoff of a contingent claim. Therefore, this is understood as an optimal investment problem of a *buyer* of the claim B, who faces the model uncertainty.

Convex duality theory is a general framework of solving optimization problems by passing to another (often easier) optimization on the dual space. There are a lot of duality theories, in diverse problems involving stochastic/functional analysis, and we establish a variant of those required to solve the robust utility maximization problem (1.1). The basic idea is described as follows. Let V be the conjugate of U, i.e.,

$$V(y) = \sup_x (U(x) - xy), \quad \forall y > 0.$$

Then some formal calculation suggests that the next inequality holds:

$$\text{(1.2)} \quad \sup_{\theta\in\Theta}\inf_{P\in\mathcal{P}} E^P[U(\theta\cdot S_T + B)] \le \inf_{\lambda>0}\inf_{P\in\mathcal{P}}\inf_{Q\in\mathcal{M}_{loc}} E^P\left[V\left(\lambda\frac{dQ}{dP}\right) + \lambda\frac{dQ}{dP}B\right].$$

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Here $\mathcal{M}_{loc}$ denotes the set of *local martingale measures* for S, which can be *interpreted* as the *dual* of the set of stochastic integrals: $\{\theta \cdot S_T : \theta \in \Theta\}$ which is the *primal domain*. We call the RHS the *dual problem* of (1.1), and this inequality is the basis of duality theory.

From the formal inequality (1.2), we expect that the *primal* problem (1.1) can be solved via the dual. In fact, if the inequality holds as *equality*, the maximal admissible utility can be computed via the RHS, which is often easier. Furthermore, if we obtain a solution $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ to the dual problem, then a separation argument using a version of Hahn-Banach theorem and Yor's closedness theorem yield a kind *martingale representation* for the density $d\widehat{Q}/d\widehat{P}$. Then the integrand appearing in the representation will turn out to be an optimal strategy for the problem (1.1), in a suitable sense. In other words, the robust utility maximization problem is completely solved via the dual problem.

These observations are of course only heuristics, and to be a rigorous mathematics, we have to solve a number of questions. The "development of duality theory" thus consists of:

1. rigorously formulate the dual problem, and prove (1.2) as equality (duality);
2. solve the dual problem, i.e., find a minimizer $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ and characterize it;
3. and recover a solution to the primal problem from $(\hat{\lambda}, \widehat{Q}, \widehat{P})$.

In comparison to other existing research, the features of our work are summarized into the following two aspects:

1. **Unbounded endowment.** Most of existing works on robust utility maximization problem consider only the *pure investment problem*, i.e., $B \equiv 0$, excepting the studies of the case with *bounded* endowment by [5] and [7]. Taking account the endowment as the terminal payoff of a contingent claim allows us to apply the utility maximization to a valuation of claims under *model uncertainty*. More specifically, we introduce a robust version of *utility indifference price*, with a representation through the duality. This is financially important as the model uncertainty has received much attention in practice. From a mathematical point of view, adding a *bounded* endowment causes no additional difficulty, as we will see in the main text. However, an *unbounded* endowment makes the problem more subtle, raising *regularity problems* in both primal and dual problems. Roughly speaking, we will show that all kinds of *regularities* required to solve the problem (1.1) are guaranteed by imposing natural (uniform) integrability conditions involving B.

2. **Utility function on the whole real line**. Utility maximization problems (either robust or subjective) show up very different natures depending on whether $\mathrm{dom}(U) = \{x \in \mathbb{R} : U(x) > -\infty\} = \mathbb{R}$ or $= \mathbb{R}_+$. Most of existing articles on robust utility maximization focus on the latter case with the exceptions [5] and [2], while we consider the former. In this case, we faces a difficulty of choosing the admissible class Θ. It is well-known that the usual admissible class Θ_{bb} (see (2.1) below) is too small to admit an optimal strategy *even when* $\mathcal{P}$ is a singleton and $B \equiv 0$ (see e.g. [6]).
When we consider the general robust case, the situation gets even worse. In the subjective case with $\mathcal{P} = \{\mathbb{P}\}$, an optimal strategy in a suitable sense can be obtained in the class of S-integrable processes θ whose stochastic integral $\theta \cdot S$ is a *supermartingale* under all *reasonable* local martingale measures. But the *robust counterpart* of this class depends on a part of *solution* $\widehat{P}$ to the dual problem, hence is not available in advance. Furthermore, the dependence on $\widehat{P}$ implies the dependence on the endowment B, which is conceptually quite undesirable when we consider the robust utility indifference valuation. Thus the choice of admissible class is a delicate issue when we consider the robust case with utility on the whole real line.
In this thesis, we employ the class Θ_{bb} of admissible strategy as the *basic* choice, showing the duality with this class. Noting that the dual problem does not directly depend on Θ, we prove that the duality is also *stable under certain changes of admissible class*, hence the maximal admissible utility is unchanged in particular. Then we consider the existence of optimal strategy in a certain enlargement of Θ_{bb}, which keeps the maximal utility unchanged.

2. Summary of Main Results

We start from an auxiliary complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Everything is defined on this space unless otherwise mentioned. In particular, every element of $\mathcal{P}$ is assumed to be absolutely continuous w.r.t. $\mathbb{P}$. The semimartingale S is always assumed to be *locally bounded*. Also, our *basic choice* of Θ is:

$$\Theta_{bb} := \{\theta \in L(S) : \theta_0 = 0,\ \theta \cdot S \text{ is uniformly bounded from below}\}. \tag{2.1}$$

Chapter 3. We first examine our idea in a simple case where the utility function is exponential: $U(x) = -e^{-\alpha x}$. This *nicely behaving* utility function dramatically reduce technical complexities, giving us a deep understanding of the structure of the problem.

With exponential utility, the primal and dual problems reduce respectively to:

$$\text{(2.2)} \qquad \text{minimize} \quad \sup_{P \in \mathcal{P}} E^P \left[e^{-\alpha(\theta \cdot S_T + B)} \right], \quad \text{among } \theta \in \Theta,$$

$$\text{(2.3)} \qquad \text{minimize} \quad \mathcal{H}(Q|P) + \alpha E^Q[B], \quad \text{among } (Q, P) \in \mathcal{M}_{\text{ent}} \times \mathcal{P}.$$

Here $\mathcal{H}(Q|P)$ is the *relative entropy* of Q w.r.t. P, and $\mathcal{M}_{\text{ent}}$ is the set of elements of $\mathcal{M}_{\text{loc}}$ having finite relative entropy w.r.t. some $P \in \mathcal{P}$. In this case, the duality to be shown simplifies to

$$\text{(2.4)} \qquad \inf_{\theta \in \Theta} \sup_{P \in \mathcal{P}} E^P[e^{-\alpha(\theta \cdot S_T + B)}] = e^{-\inf_{(Q,P) \in \mathcal{M}_{\text{ent}} \times \mathcal{P}} (\mathcal{H}(Q|P) + \alpha E^Q[B])}$$

Assume that, $\mathcal{P}$ is *weakly compact*, the market is free of arbitrage in that $\mathcal{M}_{\text{ent}}$ contains an element $\bar{Q} \sim \mathbb{P}$, and

(A3.3) the family $\{e^{-\alpha B} dP/d\mathbb{P}\}_{P \in \mathcal{P}}$ is *uniformly integrable*, and

$$\sup_{P \in \mathcal{P}} E^P[e^{(\alpha+\varepsilon)B^-}] < \infty \text{ and } \sup_{P \in \mathcal{P}} E^P[e^{\varepsilon B^+}] < \infty, \ \exists \varepsilon > 0.$$

Under these assumptions, we first prove that the dual problem (2.3) admits a solution $(\hat{Q}, \hat{P})$ possessing some reasonable properties, and the density $d\hat{Q}/d\hat{P}$ has a kind of *martingale representation*: $d\hat{Q}/d\hat{P} = c \cdot \exp(-\alpha(\hat{\theta} \cdot S_T + B))$, where c is a constant and $\hat{\theta}$ is a $(S, \hat{Q})$-integrable process whose stochastic integral $\hat{\theta} \cdot S$ is a $\hat{Q}$-martingale.

We next consider the duality for a special choice of Θ. Let Θ_b be the set of S-integrable predictable processes θ with $\theta_0 = 0$ such that $\theta \cdot S$ is uniformly bounded. Then we show that the duality (2.4) holds with $\Theta = \Theta_b$. This duality will turn out to be *stable*: (2.4) is invariant under changes of admissible class Θ in a certain range.

We proceed to the existence of an optimal strategy in an appropriate enlargement of Θ_b. As the problem (2.2) is of *minimax-type*, it suffices to find a *saddle point* of the map $(\theta, P) \mapsto E^P[\exp(-\alpha(\theta \cdot S_T + B)]$, and it is natural to ask if such a saddle point consists of the pair $(\hat{\theta}, \hat{P})$, where $\hat{P}$ is the P-part of a dual optimizer, and $\hat{\theta}$ is an "optimal strategy" under fixed probability $\hat{P}$, i.e., a minimizer of $\theta \mapsto E^{\hat{P}}[\exp(-\alpha(\theta \cdot S_T + B)]$. The latter is given by the integrand appearing in the representation of $d\hat{Q}/d\hat{P}$. Under an additional assumption, we verify this conjecture with a certain admissible class Θ_B, using the *variational inequality* characterizing the optimality of $\widehat{P}$.

We provide also a solvable example in a 2-dimensional Brownian factor setting, using a standard stochastic control technique.

Chapter 4. In the proof of duality (2.4) for the exponential case, we use the subjective duality equality, that is, the duality with *fixed* $P \in \mathcal{P}$. Such dualities are already available if either U is *exponential* ([4]) or the endowment B is *bounded* ([1]). These results motivate us to consider: to what degree of generality does the duality below hold?

$$\sup_{\theta \in \Theta_{bb}} E[U(\theta \cdot S_T + B)] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} E\left[V\left(\lambda \frac{dQ}{d\mathbb{P}}\right) + \lambda \frac{dQ}{d\mathbb{P}} B \right]. \tag{2.5}$$

Here $\mathcal{M}_V$ denotes the set of $Q \in \mathcal{M}_{loc}$ such that $E[V(dQ/d\mathbb{P})] < \infty$. We prove this duality for a wide class of utility functions U and *suitably integrable* B. The idea of proof is based on a refinement of [1] from a slightly different point of view. More precisely, we apply the Rockafellar theorem on *convex integral functionals* to a *random utility function* $(\omega, x) \mapsto U(x + B(\omega))$, by establishing some simple estimates for this random utility and its conjugate. This allows us to exploit Fenchel's general duality theorem.

Also, we provide a result on the existence of optimal strategy in a suitable admissible class. This part is rather expository, and many similar results are available with slight differences in assumptions (e.g. [3]).

Chapter 5. The dual problem in Chapter 3 was the minimization of the relative entropy with *unbounded penalty term* $Q \mapsto E^Q[B]$, over the product set $\mathcal{M}_{\text{ent}} \times \mathcal{P}$. This leads us to more general class of minimization problems, replacing the entropy $\mathcal{H}(Q|P)$ by other *f-divergence functionals* $f(Q|P)$ associated to a convex function f:

$$\text{minimize} \quad f(Q|P) + E^Q[B], \quad \text{among } (Q, P) \in \mathcal{Q} \times \mathcal{P}. \tag{2.6}$$

Here $\mathcal{Q}$ is a set of probabilities absolutely continuous w.r.t. $\mathbb{P}$. If $f(x) = x \log x$, the f-divergence coincides with the relative entropy. The case with $\mathcal{P}$ being a singleton and $B \equiv 0$ is classical with a lot of research, and [2] recently extends these to the case with *weakly compact* $\mathcal{P}$. It is known that the f-divergence functional is jointly weakly lower semicontinuous. Thus, as $\mathcal{P}$ is compact, the existence of minimizer of $(Q, P) \mapsto f(Q|P)$ is guaranteed if we have that some level set of $Q \mapsto \inf_{P \in \mathcal{P}} f(Q|P)$ is also compact, and this is the heart of [2]. Their argument works even for the case with a *penalty term*

$Q \mapsto E^Q[B]$ if B is bounded, since then the penalty is *continuous*, hence does not harm the regularity of the functional to be minimized.

However, the case with *unbounded* B is more subtle, since the penalty is no longer even *lower semicontinuous*. Thus, the regularity of the penalized f-divergence functional is not a priori trivial. Also, we need some estimates for this functional to apply the compactness ($\Leftrightarrow$ uniform integrability) criterion of [2]. In Chapter 5, we closely investigate the problem (2.6), giving the existence and variational characterization of a solution under an *uniform integrability* condition involving B.

Chapter 6. We finally develop a duality theory for robust utility maximization problem (1.1) for a wide class of utility functions and unbounded random endowment B, generalizing the idea of Chapter 3, and using the results of Chapters 4 and 5.

We first consider the duality:

$$\sup_{\theta\in\Theta}\inf_{P\in\mathcal{P}} E^P[U(\theta\cdot S_T + B)] = \inf_{\lambda>0}\inf_{(Q,P)\in\mathcal{M}_V\times\mathcal{P}}(V(\lambda Q|P) + \lambda E^Q[B]), \tag{2.7}$$

where $V(\cdot|\cdot)$ is the f-divergence associated to V, and $\mathcal{M}_V$ is the set of elements $Q \in \mathcal{M}_{loc}$ with $\inf_{P\in\mathcal{P}} V(Q|P) < \infty$. Under some assumptions including the compactness of $\mathcal{P}$ and an uniform integrability condition involving B, we first show that the order of "$\sup_{\theta\in\Theta}$" and "$\inf_{P\in\mathcal{P}}$" can be changed, reducing the *robust problem* to a *family of subjective problems*. Then the *robust* duality (2.7) follows if we can apply the the *subjective* duality of Chapter 4 to each $P \in \mathcal{P}$. Although this is not possible, we can take a *dense* subset of $\mathcal{P}$ on which the subjective duality holds. Then some approximation arguments prove the duality. Also, we introduce a robust version of *utility indifference price* of B, and compute it via the duality.

We next consider the dual problem:

$$\text{minimize} \quad V(\lambda Q|P) + \lambda E^Q[B], \quad \text{among } \lambda > 0,\ (Q,P) \in \mathcal{M}_V \times \mathcal{P}. \tag{2.8}$$

With a simple observation, this problem is decomposed into: the minimization of $(Q,P) \mapsto f_\lambda(Q|P) + E^Q[B]$ for each $\lambda > 0$, where $f_\lambda(x) = V(\lambda x)/\lambda$, and the minimization of the value function $v(\lambda) := \inf_{(Q,P)\in\mathcal{M}_V\times\mathcal{P}}(V(\lambda Q|P) + \lambda E^Q[B])$ in $\lambda > 0$. The first part is nothing other than the problem of Chapter 5, while the latter is easy.

Finally, we discuss the existence of optimal strategy for (1.1) in a certain enlargement of Θ_{bb}. As in the exponential case of Chapter 3, it is enough to find a saddle point of

$(\theta, P) \mapsto E^P[U(\theta \cdot S_T + B)]$, and a natural candidate is the pair $(\hat{\theta}, \widehat{P})$ consisting of the P-part of a dual optimizer $(\widehat{Q}, \widehat{P})$ and the optimal strategy under $\widehat{P}$. Under an additional assumption, we verify this via the variational inequality for $\widehat{P}$.

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